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ON RATIONALITY OF THE COGROWTH SERIES

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Abstract. The cogrowth series of a group G depends on the presentation of the group. We show that the cogrowth series of a non-empty presentation is a rational function not equal to 1 if and only if G is finite. Except for the trivial group, this property is independent of presentation.

Introduction

In this paper all presentations will be assumed to have a finite number of generators. The *cogrowth function* $\Gamma(n)$ of a finitely generated group G with a given presentation $\langle\, a_1, \dots , a_k \mid R \,\rangle$ can be defined as the number of closed paths based at the identity and without backtracking in the Cayley graph of G, i.e. the number of freely reduced words in $a_1^{\pm 1}, \dots , a_k^{\pm 1}$ that are equal to the identity in G (we also put $\Gamma(0) = 1$). Then the *cogrowth series* $\gamma(t)$ is the sum $\sum_{n=0}^{\infty} \Gamma(n)t^n$. The *cogrowth* of G is $\gamma = \limsup_{n\to\infty} \Gamma^{1/n}(n)$ and is the reciprocal of the radius of convergence of the cogrowth series. The return generating series $r(t)$ is defined in [4] and [6]. The definition is as for the cogrowth series except that we also count closed paths with backtracking (i.e. the words representing the identity don't have to be freely reduced). For definitions and properties of finite automata and regular languages, see [5].

The cogrowth series depends on the particular presentation of the group, however certain properties of the cogrowth series are group invariants. For example, Grigorchuk [2] and Cohen [1] have shown that G is an amenable group if and only if $\gamma/(2k - 1) = 1$, where k is the number of generators of the presentation. In this paper we give another such property. It is not difficult to see that the cogrowth series and return generating series of a finite group are rational. In fact, Quenell [6] proves that for a group G of finite order N, $r(t) = \frac{1}{N}\sum_{i=1}^{k} 1/(1 - \lambda_i t)$, where the λ_i are the eigenvalues of the adjacency matrix of the simple random walk on G. From this it follows that $r(t)$ is a rational function if G is finite. In this paper we prove:

Main Theorem. *The cogrowth series of a group G in a presentation with at least one generator is a rational function not equal to 1 if and only if the group is finite.*

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Proof of Main Theorem. Sufficiency. If G is a finite group, then the Cayley graph is finite and so, gives a finite automaton (the initial and final states are the identity) determining whether a freely reduced word is equal to the identity in G. We also know that the set of freely reduced words forms a regular language, since we only prohibit subwords of the forms $a_i a_i^{-1}$ and $a_i^{-1} a_i$, where a_i is a generator, i.e. finitely many subwords of finite length. It follows that the set of freely reduced words that are equal to the identity forms a regular language as well. Hence, its growth generating series (which is the cogrowth series of G) is rational and not equal to 1, since there are non-trivial relators. The sufficiency is proven. For an alternative proof see the Corollary to Statement 3.6 in [2]

Necessity. Assume that the cogrowth series $\gamma(t)$ of a presentation of a group is a rational function not equal to 1. Then $\gamma(t)$ cannot be a polynomial, since otherwise $\gamma(t) = 1$ and G is free with presentation $\langle\, a_1, \ldots, a_k \mid \,\rangle$. So, using partial fractions, we can write $\gamma(t)$ as a sum of simplest fractions over $\mathbf{C}$, i.e.

$$\gamma(t) = p(t) + \sum_{i=1}^{m} \frac{c_i}{(t-b_i)^{l_i}},$$

for some polynomial $p(t)$, non-zero complex constants c_i and b_i, positive integers l_i, and $m \geq 1$. Note that the b_i cannot be 0, since otherwise $\gamma(t)$ would have coefficients with negative powers of t. Also without loss of generality, we can assume that $|b_1|$ is minimal among $|b_i|$'s. We do not assume in this proof that the minimal $|b_i|$ is unique, though it follows from Proposition 1 of [1] or Lemma 1 of [7] that there are only two cases, which we will refer to later:

a) G has at least one relator of odd length, in which case the b_i of smallest absolute value is positive and unique; or

b) G has no relators of odd length, in which case there are exactly two b_i of smallest absolute value, and one of them is positive, one is negative.

Further, the radius of convergence of the Taylor series of $\gamma(t)$ is $|b_1|$, since b_1 is a singular point with the smallest absolute value. Thus, the cogrowth γ of G is $1/|b_1|$. The Taylor series of $\gamma(t)$ will be

$$p(t) + \sum_{i=1}^{m} \frac{(-1)^{l_i} c_i}{(l_i-1)! b_i^{2l_i-1}} \sum_{n=0}^{\infty} \frac{(n+l_i-1)!}{n!} \left(\frac{1}{b_i}\right)^n t^n.$$

This means that the cogrowth function of the presentation is

$$\Gamma(n) = \sum_{i=1}^{m} \frac{(-1)^{l_i} c_i}{(l_i-1)! b_i^{2l_i-1}} \frac{(n+l_i-1)!}{n!} \left(\frac{1}{b_i}\right)^n \quad \text{for } n > \deg(p(t)).$$

It follows that, for large n,

$$\Gamma(n) = \sum_{i=1}^{m} q_i(n) \left(\frac{1}{b_i}\right)^n,$$

where each $q_i(n)$ is a polynomial. For some choice of b_i with smallest absolute value, we will have $q_i(n) \neq 0$. For if $q_i(n) = 0$ for all such b_i, then the radius of convergence would be greater than $|b_1|$. Hence,

$$\lim_{n\to\infty} \Gamma(2n)/\gamma^{2n} \neq 0.$$

But, by Theorem 2 of [7],

$$\lim_{n\to\infty} \Gamma(2n)/\gamma^{2n} = \begin{cases} 1/N, & \text{where } N = |G|, \text{if } G \text{ is finite and we have case a),} \\ 2/N, & \text{if } G \text{ is finite and we have case b),} \\ 0, & \text{if } G \text{ is infinite.} \end{cases}$$

Thus the group is finite. □

Corollary. *The return generating series of a group is rational if and only if the group is finite.*

Proof. As was mentioned ealier, the return generating series of a finite group is rational. So, we are left to prove that the return generating series of an infinite group is not rational. By Lemma 1 of [7], the cogrowth series is related to the return generating series by the following formulas (in his Ph.D. thesis, the author provides an independent combinatorial proof of these formulas):

$$\gamma(t) = \frac{1-t^2}{1+(2k-1)t^2}\, r\left(\frac{t}{1+(2k-1)t^2}\right), \tag{1}$$

$$r(t) = \frac{1-k+k\sqrt{1-4(2k-1)t^2}}{1-4k^2t^2}\, \gamma\left(\frac{1-\sqrt{1-4(2k-1)t^2}}{2(2k-1)t}\right), \tag{2}$$

where k is the number of generators. If the group is free on at least one generator, then (2) shows that the return generating series is not rational. If the group is infinite and not free, then we will prove that its return generating series is not rational by contradiction. Suppose that the return generating series is rational. Then it follows from (1) that the cogrowth series is rational, which contradicts the Main Theorem. □

The results of this paper are part of the author's Ph. D. thesis, written under the direction of Dr. S. P. Humphries.

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