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A GENERALIZATION OF 2-HOMOGENEOUS CONTINUA BEING LOCALLY CONNECTED

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ABSTRACT. An elementary proof is given that if each pair of points of a homogeneous metric continuum can be mapped by a homeomorphism into an arbitrarily small connected set, then the continuum is locally connected.

In [3], Ungar answered a question of Burgess [1] by showing that every 2-homogeneous metric continuum is locally connected. Ungar's proof uses Theorem 2.1 of [2], a powerful result of Effros. This note gives a short, elementary proof which significantly generalizes Ungar's result without using the Effros theorem.

Lemma 1. *If X is a space with a countable base such that for all $x, y \in X$, there is a compact connected set C containing x and y, and an open set U containing C such that the component of U containing C is nowhere dense in X, then X is first category.*

Proof. Let $p \in X$. Since X has a countable base, it follows that there is a base $U_1, U_2, \ldots$ which is closed with respect to taking finite unions. Let $V_1, V_2, \ldots$ be the elements of this list which contain p and in which the component containing p is nowhere dense in X. Let C_i be the component of V_i which contains p. We claim that the C_i cover X.

Let $x \in X$. Then there is a continuum C in X containing p and x, and an open set U containing C such that the component of U containing C is nowhere dense in X. There exist $U_{n_1}, U_{n_2}, \ldots, U_{n_t}$ covering C such that $U_{n_i} \subseteq U$ for each i. Now $p \in U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_t}$, and the component of this set containing p is nowhere dense in X. Thus, $U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_t} = V_i$ for some i, and so $x \in C_i$. □

Lemma 2. *If X is a homogeneous, complete metric space that is not locally connected, then X has a base of open sets, each component of which is nowhere dense in X.*

Proof. Since X is homogeneous, it suffices to show that there is a nonempty open set V, each component of which is nowhere dense in X. Suppose there is no such set. Let U_1 be some particular nonempty open set of diameter < 1. Then U_1 has a component C_1 such that $(\overline{C_1})^\circ$ is nonempty. There is a nonempty open set U_2 of diameter $< 1/2$ such that $\overline{U_2} \subseteq (\overline{C_1})^\circ$. Again, U_2 must have a component C_2 such $(\overline{C_2})^\circ$ is nonempty. Continuing in this manner, one finds connected sets $\overline{C_i}$

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that form a neighborhood base at some point p in X. Since X is homogeneous, it follows that X is locally connected, contrary to the hypothesis. □

Theorem 1. *If X is a homogeneous Polish (separable, complete metric) space such that for each pair of points $x, y \in X$ there is a point $p \in X$ such that for every $\epsilon > 0$ there exists a homeomorphism $f : X \to X$ and a continuum D contained in the ϵ-neighborhood of p such that $f(x), f(y) \in D$, then X is locally connected.*

Proof. Since X is separable metric, it has a countable base. Suppose X is not locally connected; then X has a base of open sets, each component of which is nowhere dense in X. We will show that the remaining hypothesis of Lemma 1 is fulfilled, contradicting that X is second category.

Let $x, y \in X$, and let p be as above. Then there is an $\epsilon > 0$ such that each component of $B(p, \epsilon)$, the ϵ-ball centered at p, is nowhere dense in X. By hypothesis, there is a homeomorphism $f : X \to X$ such that $f(x)$ and $f(y)$ lie in a continuum D contained in $B(p, \epsilon)$. The sets $C = f^{-1}(D)$ and $U = f^{-1}(B(p, \epsilon))$ fulfill the requirements of Lemma 1. □

Theorem 2. *If X is a compact metric homogeneous space such that each pair of points can be mapped by homeomorphisms from X to X into connected sets of arbitrarily small diameter, then X is locally connected.*

Proof. Let $x, y \in X$. The proof is as above except that in this case if X is not locally connected we can cover X with finitely many open sets, each component of which is nowhere dense in X. Utilizing a Lebesgue number for such a cover, it easily follows from the hypothesis that there is a homeomorphism $f : X \to X$ such that $f(x)$ and $f(y)$ lie in a continuum D contained in an open set V in this cover. The sets $C = f^{-1}(D)$ and $U = f^{-1}(V)$ fulfill the requirements of Lemma 1. □

A space X is 2-homogeneous if for all points $a, b, c, d \in X$, with $a \neq b$ and $c \neq d$, there is a homeomorphism from X to itself which carries the set $\{a, b\}$ into the set $\{c, d\}$. If x and y are distinct elements of a 2-homogeneous space, z is a third point, and f is a homeomorphism carrying $\{x, z\}$ into $\{y, z\}$, then either f or f^2 maps x to y; thus, every 2-homogeneous continuum is homogeneous. Since nondegenerate metric continua always have nondegenerate connected subsets of small diameter, Ungar's result immediately follows.

Corollary ([3, 3.12])**.** *Every 2-homogeneous metric continuum is locally connected.*

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