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HOMOGENEOUS IDEALS IN WICK $*$-ALGEBRAS

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Abstract. The necessary and sufficient condition for the family of homogeneous elements to determine a Wick ideal is presented. The structure of homogeneous Wick ideals with degree higher than 2 is discussed. For the braided operator T a formula to calculate the largest cubic ideal when the quadratic one is known is obtained. Irreducible $*$-representations of the μ-CAR algebra are classified.

Introduction

In the study of $*$-representations of Wick algebras, it is useful to know the structure of Wick ideals, and especially, as examples show [1], of homogeneous Wick ideals. For example, in any bounded representation of μ-CCR or q_{ij}-CCR, $|q_{ij}| = 1$ as $i \neq j$, any quadratic ideal vanishes.

In the paper [1], the main attention was concentrated on quadratic ideals, and there was presented a necessary and sufficient condition imposed on the system of elements to generate a quadratic ideal.

In this note this criterion is extended to the general case (Sec. 2). In the case when the operator T (see Sec. 1) satisfies the braid relation, and $-1 \leq T \leq 1$, we investigate the structure of homogeneous Wick ideals of higher degrees. In particular, we improve the theorem on strict positivity of the Fock representation, and derive a formula, which allows one to calculate a cubic ideal providing that a quadratic one is known (Sec. 3). In Sec. 4 these results are illustrated in the example of μ-CAR algebra.

1. Preliminaries

Let $I = \{1, \dots, d\}$, and $T_{ij}^{kl} \in \mathbb{C}$, $i, j, k, l \in I$, be such that $T_{ij}^{kl} = \bar{T}_{ji}^{lk}$. The Wick algebra with coefficients $\{T_{ij}^{kl}\}$ (see [1]) is a $*$-algebra generated by the elements a_i, a_i^* and the defining relations

$$a_i^* a_j = \delta_{ij} 1 + \sum_{k,l=1}^{d} T_{ij}^{kl} a_l a_k^*.$$

Denote by $\mathcal{H} = \langle e_1, \dots, e_d \rangle$ the finite-dimensional space over $\mathbb{C}$, and by $\mathcal{H}^*$ its formal dual. $\mathcal{T}(\mathcal{H}, \mathcal{H}^*)$ will denote the tensor algebra over $\mathcal{H}$, $\mathcal{H}^*$. Then $\mathcal{W}$ can be

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canonically realized as

$$\mathcal{T}(\mathcal{H},\mathcal{H}^*)\Big/\Big\langle e_i^*\otimes e_j-\delta_{ij}1-\sum T_{ij}^{kl}e_l\otimes e_k^*\Big\rangle.$$

In this realization, the subalgebra, generated by $\{a_i\}$ is identified with $\mathcal{T}(\mathcal{H})$.

It is obvious that any element of $\mathcal{W}$ can be uniquely represented as a polynomial in the noncommuting variables a_i, a_i^*, where in each monomial, variables a_i are placed to the left from a_j^*. Such monomials are called Wick ordered ones, and they form a basis in $\mathcal{W}$.

When studying properties of $\mathcal{W}$, one can find useful the following operators (see [1]):

$$\begin{aligned}
&T\colon \mathcal{H}\otimes\mathcal{H}\mapsto\mathcal{H}\otimes\mathcal{H},\quad Te_k\otimes e_l=\sum_{i,j}T_{ik}^{lj}e_i\otimes e_j,\\
&T_i\colon \mathcal{H}^{\otimes n}\mapsto\mathcal{H}^{\otimes n},\quad T_i=\underbrace{1\otimes\cdots\otimes 1}_{i-1}\otimes T\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-i-1},\\
&R_n\colon \mathcal{H}^{\otimes n}\mapsto\mathcal{H}^{\otimes n},\quad R_n=1+T_1+T_1T_2+\cdots+T_1T_2\cdots T_{n-1},\\
&P_n\colon \mathcal{H}^{\otimes n}\mapsto\mathcal{H}^{\otimes n},\quad P_2=R_2,\ P_{n+1}=(1\otimes P_n)R_{n+1}.
\end{aligned}$$

In what follows, we will use the commutation rule in $\mathcal{W}$ between e_i^* and X, where $X\in\mathcal{H}^{\otimes n}$.

Proposition 1. *Let $X\in\mathcal{H}^{\otimes n}$; then*

$$e_i^*\otimes X=\mu(e_i^*)R_nX+\mu(e_i^*)\sum_{k=1}^{d}T_1T_2\cdots T_n(X\otimes e_k)e_k^*, \tag{1}$$

where $\mu(e_i^*)\colon\mathcal{T}(\mathcal{H})\mapsto\mathcal{T}(\mathcal{H})$ is defined as follows:

$$\mu(e_i^*)1=0,\quad \mu(e_i^*)e_{i_1}\otimes\cdots\otimes e_{i_n}=\delta_{ii_1}e_{i_2}\otimes\cdots\otimes e_{i_n}.$$

Proof. It follows from the defining relations that $e_i^*\otimes X\in\mathcal{H}^{\otimes n-1}\oplus\mathcal{H}^{\otimes n}\otimes\mathcal{H}^*$. The fact that the term $e_i^*\otimes X$ from $\mathcal{H}^{\otimes n-1}$ is equal to the first term in the formula (1) is essentially contained in [1], Lemma 2.1.1.

Now we prove that the term belonging to $\mathcal{H}^{\otimes n}\otimes\mathcal{H}^*$ is equal to the second term. It is obvious that one can assume $X=e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_n}$. Denote by F_i the component of $e_i^*\otimes X$ which belongs to $\mathcal{H}^{\otimes n}\otimes\mathcal{H}^*$. Consider "deriving function" $F=\sum_{i=1}^d e_i\otimes F_i$. It is evident that $F_i=\mu(e_i^*)F$. The relations imply

$$F=\sum_i\sum_{k_1,l_1}\cdots\sum_{k_n,l_n}T_{ii_1}^{k_1l_1}T_{k_1i_2}^{k_2l_2}\cdots T_{k_{n-1}i_n}^{k_nl_n}e_i\otimes e_{l_1}\otimes\cdots\otimes e_{l_n}\otimes e_{k_n}^*.$$

Taking into account the definition of T_i and changing summation order, we get

$$F=\sum_{k_n}T_1\cdots T_n(e_{i_1}\otimes\cdots\otimes e_{i_n}\otimes e_{k_n})\otimes e_{k_n}^*,$$

which completes the proof. □

2. Homogeneous ideals. General case

A Wick ideal (see [1]) is a two-sided ideal $I\subset\mathcal{T}(\mathcal{H})$, such that $\mathcal{T}(\mathcal{H}^*)\subset I\mathcal{T}(\mathcal{H}^*)$. If I is generated by a set $I_0\subset\mathcal{H}^{\otimes n}$, then I is called a homogeneous Wick ideal of degree n.

The following statement is a generalization of the fact established in [1] for $n=2$.

Proposition 2. *Let $P\colon \mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}$ be a projection. Then $I_n = \langle P\mathcal{H}^{\otimes n}\rangle$ is a Wick ideal if and only if*

1. $R_n P = 0$,
2. $[1 \otimes (1-P)]T_1T_2\cdots T_n[P\otimes 1] = 0$.

Moreover, if T satisfies the braid condition $T_1T_2T_1 = T_2T_1T_2$ and P is a projection on $\ker R_n$, then the condition 2 holds automatically.

Proof. It follows from [1], Lemma 3.1.1 that P generates a homogeneous ideal if and only if $\forall i = 1, \dots, d$, $\forall X \in \mathcal{H}^{\otimes n}$, the relation

$$e_i^* \otimes PX \in P(\mathcal{H}^{\otimes n}) \otimes \mathcal{H}^*$$

holds.

Then we have

$$e_i^* \otimes PX = \mu(e_i^*)\Big(R_nPX + \sum_{k=1}^{d} T_1\cdots T_n(PX\otimes e_k)\otimes e_k^*\Big).$$

Since $\mu(e_i^*)R_nPX \in \mathcal{H}^{\otimes(n-1)}$, we have $\mu(e_i^*)R_nPX = 0$ $\forall i, X$, which implies $R_nP = 0$. Further, $\mu(e_i^*)\sum T_1\cdots T_n(PX\otimes e_k)\otimes e_k^* \in P(\mathcal{H}^{\otimes n})\otimes\mathcal{H}^*$ if and only if $\forall k$ $\mu(e_i^*)T_1\cdots T_n(PX\otimes e_k) \in P(\mathcal{H}^{\otimes n})$, i.e., if

$$(1-P)\mu(e_i^*)T_1\cdots T_n(P\otimes 1)(X\otimes e_k) = 0,$$
$$\mu(e_i^*)(1\otimes(1-P))T_1\cdots T_n(P\otimes 1)(X\otimes e_k) = 0.$$

Since the equality holds $\forall i, k, X$, we have

$$[1\otimes(1-P)]T_1\cdots T_n[P\otimes 1] = 0.$$

Now, let T satisfy the braid condition. Then it is easy to check that $\forall k = 1, \dots, n-1$ we have $T_1T_2\cdots T_nT_k = T_{k+1}T_1T_2\cdots T_n$, which implies

$$T_1T_2\cdots T_n(R_n\otimes 1) = (1\otimes R_n)T_1T_2\cdots T_n.$$

Therefore, if $X \in \ker R_n$, then $\forall k$ we have

$$(1\otimes R_n)T_1\cdots T_n(X\otimes e_k) = T_1\cdots T_n(R_n\otimes 1)(X\otimes e_k) = 0.$$

The proof is done. □

3. Homogeneous ideals. Braided case

In this section, we assume that T satisfies the braid relation, and $-1 \le T \le 1$. Under these conditions, the maximal quadratic ideal I_2 is generated by $\ker(1+T)$. It is obvious that if $-1 < T \le 1$, then $I_2 = \{0\}$. It appears that in this case $\forall n \ge 2$, $I_n = \{0\}$.

Proposition 3. *If $-1 < T \le 1$, and $T_1T_2T_1 = T_2T_1T_2$, then $\ker R_n = \{0\}$ for all $n \ge 2$.*

Remark 1. This statement is an elaboration of Theorem 2.3 in [2] for the case S_n.

If $-1 \le T < 1$, then one can prove that $I_n \subset I_2$. For $n = 3$, a more precise statement holds.

Theorem 1. *If $T_1T_2T_1 = T_2T_1T_2$ and $-1 \le T \le 1$, then*

$$\ker R_3 = (1 - T_1T_2)(\ker R_2 \otimes \mathcal{H});$$

in particular, $I_3 \subset I_2$.

Proof. We recall that $R_2 = 1 + T$; then $\ker R_2 \otimes \mathcal{H} = \ker(1 + T_1)$, $\mathcal{H} \otimes \ker R_2 = \ker(1 + T_2)$. Consider operator $P_3 = (1 \otimes R_2)R_3$; then it is obvious that $\ker R_3 \subset \ker P_3$. We prove that $\ker P_3 = \ker(1 + T_1) + \ker(1 + T_2)$. To do it, we need the following formulas:

$$P_3 = (1 + T_2)(1 + T_1 + T_1T_2),$$
$$P_3 = (1 + T_1)(1 + T_2 + T_2T_1),$$
$$R_3 = 1 + T_1 + T_1T_2,\ \hat{R}_3 = 1 + T_2 + T_2T_1.$$

These formulas demonstrate that R_3, $\hat{R}_3\colon \ker P_3 \mapsto \ker P_3$, and R_3 maps $\ker P_3$ on $\ker(1 + T_2)$, $\hat{R}_3$ projects on $\ker(1 + T_1)$. We denote by R the restriction of the operator $R_3 + \hat{R}_3$ to $\ker P_3$. Since $R = R^*$, we have $\ker P_3 = \operatorname{im} R \oplus \ker R$, and moreover $\operatorname{im} R \subset \ker(1 + T_1) + \ker(1 + T_2)$. Let $Y \in \ker\ R$. Since $R_3 + \hat{R}_3 = P_3 + 1 - T_1T_2T_1$ and $Y \in \ker\ P_3$, we have $(1 - T_1T_2T_1)Y = 0$. The latter equality holds if and only if the following equalities hold:

$$(1 - T_1^2)Y + T_1(1 - T_2)T_1Y = 0,$$
$$(1 - T_2^2)Y + T_2(1 - T_1)T_2Y = 0.$$

Taking into account the condition $-1 \le T \le 1$, we get

$$(1 - T_1^2)Y = 0, \quad (1 - T_2^2)Y = 0, \quad T_2T_1Y = T_1Y, \quad T_1T_2Y = T_2Y.$$

The condition $P_3Y = 0$ takes the form $(1 + T_1 + T_2)Y = 0$. Now we can easily check that

$$(1 - T_1)Y \in \ker(1 + T_1), \quad (2 + T_1Y) \in \ker(1 + T_2),$$

which implies that $Y \in \ker(1 + T_1) + \ker(1 + T_2)$.

Therefore, $\ker\ R \subset \ker(1 + T_1) + \ker(1 + T_2)$, and thus $\ker P_3 = \ker R_2 \otimes \mathcal{H} + \mathcal{H} \otimes \ker R_2$.

Since the restriction of R_3 to $\ker P_3$ is a projection, and $\ker R_3 \subset \ker P_3$, we have $\ker R_3 = (1 - R_3)(\ker P_3)$. It is easy to check that $(1 - R_3)(\ker P_3) = (1 - T_1T_2)(\ker R_2 \otimes \mathcal{H})$. □

4. $*$-Representations of μ-CAR algebra

Here we consider $*$-representations of the μ-CAR algebra by the Hilbert space operators. This algebra was introduced in [1] as the algebra generated by the elements a_i, a_i^*, $i = 1, \dots, d$, which satisfy the following relations:

$$a_i^*a_i = 1 - a_ia_i^* - (1 - \mu^2)\sum_{k<i} a_ka_k^*,$$
$$a_i^*a_j = -\mu a_ja_i^*, \qquad 0 < \mu < 1.$$

It is easy to see that any $*$-representation of μ-CAR is bounded.

To classify $*$-representations of any Wick algebra it is useful to investigate the Wick ideal structure of this algebra, and especially to describe quadratic Wick ideals (see [1]). For μ-CAR the largest quadratic ideal

$$I_2 = \langle a_i^2,\ i = 1, \dots, d;\ a_ja_i + \mu a_ia_j,\ i < j\ \rangle$$

is very large; we consider a smaller quadratic ideal

$$\hat{I}_2 = \langle a_i^2,\ i = 1, \dots, d - 1;\ a_ja_i + \mu a_ia_j,\ i < j\ \rangle.$$

The main result of this note is the following theorem:

Theorem 2. *The ideal $\hat{I}_2$ vanishes in any representation of the μ-CAR algebra.*

Proof. Let us denote $A = a_1^2$, $B = a_2 a_1 + \mu a_1 a_2$. It follows from the basic relations that

$$\begin{aligned} A^* A &= A A^*, \\ A^* a_k &= \mu^2 a_k A^*, \qquad k > 1. \end{aligned}$$

By the Fuglede-Phutnam theorem we have that $A a_k = \mu^2 a_k A$, $k > 1$. (It is obvious that $A a_1 = a_1 A$.) It is easy to see, that these relations imply that in any irreducible representation either $A = 0$ or $\ker A = \{0\}$. Suppose now that $\ker A = \{0\}$. Then we have

$$\begin{aligned} B^* B &= \mu^2 B B^* + (1 - \mu^4)(1 + \mu^2) A A^*, \\ A^* B &= \mu^2 B A^*, \quad A B = \mu^2 B A. \end{aligned}$$

Since $A A^* > 0$, we have $B^* B > 0$. Let π be an irreducible representation of μ-CAR. Consider a polar decomposition $\pi(b^*) = WT$; here $T \geq 0$ and W is coisometry. Using the unitary reduction we can represent $\pi(A)$, T, W^* in the following form:

$$\begin{aligned} \pi(A) &= \operatorname{diag}(\lambda \mu^{2n} x 1,\ n = 0, 1, 2, \dots), \\ T &= \operatorname{diag}(T_n,\ n = 0, 1, 2, \dots), \\ T_0 &= 0, \\ T_n &= x(1 + \mu^2) \mu^{n-1} (1 - \mu^{2n})^{\frac{1}{2}}, \qquad n \geq 1, \end{aligned}$$

W^* is multiple of the unilateral shift. Since $a_1 A = A a_1$, $a_1^* A = A a_1^*$, we have

$$a_1 = \operatorname{diag}(b_i,\ i = 1, 2, \dots),$$

$B = T W^*$, and in the matrix form:

$$B = \begin{pmatrix} 0 & & & \\ T_1 & 0 & & \\ & T_2 & 0 & \\ & & \ddots & \ddots \end{pmatrix}.$$

The relation $a_1^* B = \mu B a_1^*$ takes the form:

$$b_{n+1}^* T_n = \mu T_n b_n^*.$$

If $x \neq 0$, then $b_k = \mu^{k-1} b_1$, and from $a_1^* a_1 = 1 - a_1 a_1^*$ we have:

$$\begin{aligned} b_1^* b_1 &= 1 - b_1 b_1^*, \\ \mu^2 b_1 b_1^* &= 1 - \mu^2 b_1 b_1^*. \end{aligned}$$

The latter equations are compatible only in the case $\mu^2 = 1$. This implies that $x = 0$, $B = 0$, $A = 0$.

Let us denote $B_k = a_k a_1 + \mu a_1 a_k$, $k > 2$; then from the basic relations we have:

$$B_k^* B_k = \mu^2 B_k B_k^* + \mu^2 (1 - \mu^2) \sum_{1 < i < k} B_i B_i^* + (1 + \mu^2)(1 - \mu^4) A A^*.$$

It follows from the boundedness of B_k that applying induction we obtain $B_k = 0$, $k = 3, \dots, d$. Hence we have $a_k a_1 + \mu a_1 a_k = 0$, and in the irreducible representation

$$a_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes 1$$
$$a_k = \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix} \otimes \hat{a}_k, \quad k > 1.$$

where $\{\hat{a}_k, k > 1\}$ satisfy μ-CAR with $d-1$ generators.

It is obvious that $a_j^2 = 0$ if and only if $\hat{a}_j^2 = 0$ and $a_j a_i + \mu a_i a_j = 0$ if and only if its relation holds for $\hat{a}_j$, $\hat{a}_i$. Then the induction on d completes the proof. $\square$

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