HOMOGENEOUS IDEALS IN WICK $\ast$-ALGEBRAS

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Abstract. The necessary and sufficient condition for the family of homogeneous elements to determine a Wick ideal is presented. The structure of homogeneous Wick ideals with degree higher than 2 is discussed. For the braided operator $T$ a formula to calculate the largest cubic ideal when the quadratic one is known is obtained. Irreducible $\ast$-representations of the $\mu$-CAR algebra are classified.

Introduction

In the study of $\ast$-representations of Wick algebras, it is useful to know the structure of Wick ideals, and especially, as examples show [1], of homogeneous Wick ideals. For example, in any bounded representation of $\mu$-CCR or $q_{ij}$-CCR, $|q_{ij}| = 1$ as $i \neq j$, any quadratic ideal vanishes.

In the paper [1], the main attention was concentrated on quadratic ideals, and there was presented a necessary and sufficient condition imposed on the system of elements to generate a quadratic ideal.

In this note this criterion is extended to the general case (Sec. 2). In the case when the operator $T$ (see Sec. 1) satisfies the braid relation, and $-1 \leq T \leq 1$, we investigate the structure of homogeneous Wick ideals of higher degrees. In particular, we improve the theorem on strict positivity of the Fock representation, and derive a formula, which allows one to calculate a cubic ideal providing that a quadratic one is known (Sec. 3). In Sec. 4 these results are illustrated in the example of $\mu$-CAR algebra.

1. Preliminaries

Let $I = \{1, \ldots, d\}$, and $T_{ij}^{kl} \in \mathbb{C}$, $i, j, k, l \in I$, be such that $T_{ij}^{kl} = T_{ji}^{lk}$. The Wick algebra with coefficients $\{T_{ij}^{kl}\}$ (see [1]) is a $\ast$-algebra generated by the elements $a_i$, $a_i^\ast$ and the defining relations

$$a_i^\ast a_j = \delta_{ij}1 + \sum_{k,l=1}^{d} T_{ij}^{kl} a_l a_k^\ast.$$

Denote by $\mathcal{H} = \langle e_1, \ldots, e_d \rangle$ the finite-dimensional space over $\mathbb{C}$, and by $\mathcal{H}^*$ its formal dual. $T(\mathcal{H}, \mathcal{H}^*)$ will denote the tensor algebra over $\mathcal{H}$, $\mathcal{H}^*$. Then $\mathcal{V}$ can be
canonically realized as
\[ T(\mathcal{H}, \mathcal{H}^*) \bigg/ \left\langle e_i^* \otimes e_j - \delta_{ij} 1 - \sum T^{k}_{ij} e_k \otimes e_k^* \right\rangle. \]

In this realization, the subalgebra, generated by \( \{a_i\} \) is identified with \( T(\mathcal{H}) \).

It is obvious that any element of \( \mathcal{W} \) can be uniquely represented as a polynomial in the noncommuting variables \( a_i, \ a_i^* \), where in each monomial, variables \( a_i \) are placed to the left from \( a_i^* \). Such monomials are called Wick ordered ones, and they form a basis in \( \mathcal{W} \).

When studying properties of \( \mathcal{W} \), one can find useful the following operators (see [1]):

\( T : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}, \ T e_k \otimes e_l = \sum_{i,j} T^{k}_{ij} e_i \otimes e_j, \)

\( T_i : \mathcal{H}^\otimes n \rightarrow \mathcal{H}^\otimes n, \ T_i = \sum_{i=1}^{n-1} 1 \otimes T \otimes 1 \otimes \cdots \otimes 1, \)

\( R_n : \mathcal{H}^\otimes n \rightarrow \mathcal{H}^\otimes n, \ R_n = 1 + T_1 + T_1 T_2 + \cdots + T_1 T_2 \cdots T_{n-1}, \)

\( P_n : \mathcal{H}^\otimes n \rightarrow \mathcal{H}^\otimes n, \ P_2 = R_2, \ P_{n+1} = (1 \otimes P_n) R_n + 1. \)

In what follows, we will use the commutation rule in \( \mathcal{W} \) between \( e_i^* \) and \( X \), where \( X \in \mathcal{H}^\otimes n \).

**Proposition 1.** Let \( X \in \mathcal{H}^\otimes n \); then

\[ e_i^* \otimes X = \mu(e_i^*) R_n X + \mu(e_i^*) \sum_{k=1}^{d} T_j T_2 \cdots T_n (X \otimes e_k) e_k^*, \]

where \( \mu(e_i^*) : T(\mathcal{H}) \rightarrow T(\mathcal{H}) \) is defined as follows:

\[ \mu(e_i^*) 1 = 0, \quad \mu(e_i^*) e_{i_1} \otimes \cdots \otimes e_{i_n} = \delta_{i_1 i_2} \otimes \cdots \otimes e_{i_n}. \]

**Proof.** It follows from the defining relations that \( e_i^* \otimes X \in \mathcal{H}^\otimes n-1 \otimes \mathcal{H}^\otimes n \otimes \mathcal{H}^* \). The fact that the term \( e_i^* \otimes X \) from \( \mathcal{H}^\otimes n-1 \) is equal to the first term in the formula (1) is essentially contained in [1], Lemma 2.1.1.

Now we prove that the term belonging to \( \mathcal{H}^\otimes n \otimes \mathcal{H}^* \) is equal to the second term. It is obvious that one can assume \( X = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \). Denote by \( F_i \) the component of \( e_i^* \otimes X \) which belongs to \( \mathcal{H}^\otimes n \otimes \mathcal{H}^* \). Consider “deriving function” \( F = \sum_{i=1}^{d} e_i \otimes F_i \). It is evident that \( F_1 = \mu(e_i^*) F \). The relations imply

\[ F = \sum_{i} \sum_{k_1 l_1} \cdots \sum_{k_n l_n} T^{k_1 l_1}_{i_1 i_2} T^{k_2 l_2}_{i_2 i_3} \cdots T^{k_n l_n}_{i_{n-1} i_n} e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{k_n}^*. \]

Taking into account the definition of \( T_i \) and changing summation order, we get

\[ F = \sum_{k_n} T_1 \cdots T_n (e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{k_n}) \otimes e_{k_n}^*, \]

which completes the proof. \( \square \)

2. Homogeneous ideals. General case

A Wick ideal (see [1]) is a two-sided ideal \( I \subset T(\mathcal{H}) \), such that \( T(\mathcal{H}^*) \subset IT(\mathcal{H}^*) \). If \( I \) is generated by a set \( I_0 \subset \mathcal{H}^\otimes n \), then \( I \) is called a homogeneous Wick ideal of degree \( n \).

The following statement is a generalization of the fact established in [1] for \( n = 2 \).
Proposition 2. Let $P: \mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}$ be a projection. Then $I_n = \langle PH^{\otimes n} \rangle$ is a Wick ideal if and only if

1. $R_n P = 0$,
2. $[1 \otimes (1 - P)]T_1 T_2 \cdots T_n [P \otimes 1] = 0$.

Moreover, if $T$ satisfies the braid condition $T_1 T_2 T_1 = T_2 T_1 T_2$ and $P$ is a projection on $\ker R_n$, then the condition 2 holds automatically.

Proof. It follows from [1], Lemma 3.1.1 that $P$ generates a homogeneous ideal if and only if $\forall i = 1, \ldots, d, \forall X \in \mathcal{H}^{\otimes n}$, the relation

$$e_i^* \otimes PX \in P(\mathcal{H}^{\otimes n}) \otimes \mathcal{H}^*$$

holds.

Then we have

$$e_i^* \otimes PX = \mu(e_i^*) \left( R_n PX + \sum_{k=1}^{d} T_1 \cdots T_n (PX \otimes e_k) \otimes e_i^* \right).$$

Since $\mu(e_i^*) R_n PX \in \mathcal{H}^{\otimes (n-1)}$, we have $\mu(e_i^*) R_n PX = 0 \forall i, X$, which implies $R_n P = 0$. Further, $\mu(e_i^*) \sum T_1 \cdots T_n (PX \otimes e_k) \otimes e_i^* \in P(\mathcal{H}^{\otimes n}) \otimes \mathcal{H}^*$ if and only if $\forall k \mu(e_i^*) T_1 \cdots T_n (PX \otimes e_k) \in P(\mathcal{H}^{\otimes n})$, i.e., if

$$\begin{align*}
(1 - P) \mu(e_i^*) T_1 \cdots T_n (P \otimes 1)(X \otimes e_k) &= 0, \\
\mu(e_i^*) (1 \otimes (1 - P)) T_1 \cdots T_n (P \otimes 1)(X \otimes e_k) &= 0.
\end{align*}$$

Since the equality holds $\forall k, i, X$, we have

$$[1 \otimes (1 - P)]T_1 \cdots T_n [P \otimes 1] = 0.$$

Now, let $T$ satisfy the braid condition. Then it is easy to check that $\forall k = 1, \ldots, n - 1$ we have $T_1 T_2 \cdots T_n T_k = T_k T_1 T_2 \cdots T_n$, which implies

$$T_1 T_2 \cdots T_n (R_n \otimes 1) = (1 \otimes R_n) T_1 T_2 \cdots T_n.$$

Therefore, if $X \in \ker R_n$, then $\forall k$ we have

$$\begin{align*}
(1 \otimes R_n) T_1 \cdots T_n (X \otimes e_k) &= T_1 \cdots T_n (R_n \otimes 1)(X \otimes e_k) = 0.
\end{align*}$$

The proof is done. \qed

3. Homogeneous ideals. Braided case

In this section, we assume that $T$ satisfies the braid relation, and $-1 \leq T \leq 1$.

Under these conditions, the maximal quadratic ideal $I_2$ is generated by $\ker(1 + T)$. It is obvious that if $-1 < T \leq 1$, then $I_2 = \{0\}$. It appears that in this case $\forall n \geq 2$, $I_n = \{0\}$.

Proposition 3. If $-1 < T \leq 1$, and $T_1 T_2 T_1 = T_2 T_1 T_2$, then $\ker R_n = \{0\}$ for all $n \geq 2$.

Remark 1. This statement is an elaboration of Theorem 2.3 in [2] for the case $S_n$.

If $-1 \leq T < 1$, then one can prove that $I_n \subset I_2$. For $n = 3$, a more precise statement holds.

Theorem 1. If $T_1 T_2 T_1 = T_2 T_1 T_2$ and $-1 \leq T \leq 1$, then

$$\ker R_3 = (1 - T_1 T_2)(\ker R_2 \otimes \mathcal{H});$$

in particular, $I_3 \subset I_2$. 
\textit{Proof.} We recall that \( R_2 = 1 + T \); then \( \ker R_2 \otimes \mathcal{H} = \ker(1 + T_1), \mathcal{H} \otimes \ker R_2 = \ker(1 + T_2) \). Consider operator \( P_3 = (1 \otimes R_2)R_3 \); then it is obvious that \( \ker R_3 \subset \ker P_3 \). We prove that \( \ker P_3 = \ker(1 + T_1) + \ker(1 + T_2) \). To do it, we need the following formulas:

\[
P_3 = (1 + T_2)(1 + T_1 + T_1T_2),
\]
\[
P_3 = (1 + T_1)(1 + T_2 + T_2T_1),
\]
\[
R_3 = 1 + T_1 + T_1T_2, \quad \hat{R}_3 = 1 + T_2 + T_2T_1.
\]

These formulas demonstrate that \( R_3, \hat{R}_3 \colon \ker P_3 \hookrightarrow \ker P_3 \), and \( R_3 \) maps \( \ker P_3 \) on \( \ker(1 + T_2) \), \( \hat{R}_3 \) projects on \( \ker(1 + T_1) \). We denote by \( R \) the restriction of the operator \( R_3 + \hat{R}_3 \) to \( \ker P_3 \). Since \( R = R^* \), we have \( \ker P_3 = \text{im} R \oplus \ker R \), and moreover \( \text{im} R \subset \ker(1 + T_1) + \ker(1 + T_2) \). Let \( Y \in \ker R \). Since \( R_3 + \hat{R}_3 = P_3 + 1 - T_1T_2T_1 \) and \( Y \in \ker P_3 \), we have \( (1 - T_1T_2T_1)Y = 0 \). The latter equality holds if and only if the following equalities hold:

\[
(1 - T_1^2)Y + T_1(1 - T_2)T_1Y = 0,
\]
\[
(1 - T_2^2)Y + T_2(1 - T_1)T_2Y = 0.
\]

Taking into account the condition \(-1 \leq T \leq 1\), we get

\[
(1 - T_1^2)Y = 0, \quad (1 - T_2^2)Y = 0, \quad T_2T_1Y = T_1Y, \quad T_1T_2Y = T_2Y.
\]

The condition \( P_3Y = 0 \) takes the form \( (1 + T_1 + T_2)Y = 0 \). Now we can easily check that

\[
(1 - T_1)Y \in \ker(1 + T_1), \quad (2 + T_1Y) \in \ker(1 + T_2),
\]

which implies that \( Y \in \ker(1 + T_1) + \ker(1 + T_2) \).

Therefore, \( \ker R \subset \ker(1 + T_1) + \ker(1 + T_2) \), and thus \( \ker P_3 = \ker R_2 \otimes \mathcal{H} + \mathcal{H} \otimes \ker R_2 \).

Since the restriction of \( R_3 \) to \( \ker P_3 \) is a projection, and \( \ker R_3 \subset \ker P_3 \), we have \( \ker R_3 = (1 - R_3)(\ker P_3) \). It is easy to check that \( (1 - R_3)(\ker P_3) = (1 - T_1T_2)(\ker R_2 \otimes \mathcal{H}) \).

\[\square\]

4. \( * \)-Representations of \( \mu \)-CAR algebra

Here we consider \( * \)-representations of the \( \mu \)-CAR algebra by the Hilbert space operators. This algebra was introduced in [1] as the algebra generated by the elements \( a_i, a_i^*, i = 1, \ldots, d \), which satisfy the following relations:

\[
a_i^*a_i = 1 - a_i^*a_i - (1 - \mu^2) \sum_{k<i} a_k^*a_k,
\]
\[
a_i^*a_j = -\mu a_ja_i^*, \quad 0 < \mu < 1.
\]

It is easy to see that any \( * \)-representation of \( \mu \)-CAR is bounded.

To classify \( * \)-representations of any Wick algebra it is useful to investigate the Wick ideal structure of this algebra, and especially to describe quadratic Wick ideals (see [1]). For \( \mu \)-CAR the largest quadratic ideal

\[
I_2 = \langle a_i^2, i = 1, \ldots, d; a_ja_i + \mu a_i a_j, i < j \rangle
\]

is very large; we consider a smaller quadratic ideal

\[
\hat{I}_2 = \langle a_i^2, i = 1, \ldots, d - 1; a_ja_i + \mu a_i a_j, i < j \rangle.
\]

The main result of this note is the following theorem:
Theorem 2. The ideal $\hat{I}_2$ vanishes in any representation of the $\mu$-CAR algebra.

Proof. Let us denote $\tilde{A} = a_1^2$, $B = a_2a_1 + \mu a_1a_2$. It follows from the basic relations that

\begin{align*}
A^*A &= AA^*, \\
A^*a_k &= \mu^2a_kA^*, \quad k > 1.
\end{align*}

By the Fuglede-Phutnam theorem we have that $Aa_k = \mu^2a_kA$, $k > 1$. (It is obvious that $Aa_1 = a_1A$.) It is easy to see, that these relations imply that in any irreducible representation either $A = 0$ or $\ker A = \{0\}$. Suppose now that $\ker A = \{0\}$. Then we have

\begin{align*}
B^*B &= \mu^2BB^* + (1 - \mu^2)(1 + \mu^2)AA^*, \\
A^*B &= \mu^2BA^*, \quad AB = \mu^2BA.
\end{align*}

Since $AA^* > 0$, we have $B^*B > 0$. Let $\pi$ be an irreducible representation of $\mu$-CAR. Consider a polar decomposition $\pi(b^*) = WT$; here $T \geq 0$ and $W$ is coisometry. Using the unitary reduction we can represent $\pi(A), T, W^*$ in the following form:

\begin{align*}
\pi(A) &= \text{diag}(\lambda \mu^{2n}x_1, n = 0, 1, 2, \ldots), \\
T &= \text{diag}(T_n, n = 0, 1, 2, \ldots), \\
T_0 &= 0, \\
T_n &= x(1 + \mu^2)\mu^{n-1}(1 - \mu^{2n})^{\frac{1}{2}}, \quad n \geq 1,
\end{align*}

$W^*$ is multiple of the unilateral shift. Since $a_1A = Aa_1$, $a_1^*A = Aa_1^*$, we have

\begin{align*}
a_1 &= \text{diag}(b_i, i = 1, 2, \ldots),
\end{align*}

$B = TW^*$, and in the matrix form:

\begin{align*}
B &= \begin{pmatrix}
0 & 0 & \cdots \\
T_1 & 0 & \cdots \\
T_2 & 0 & \cdots \\
& \ddots & \ddots
\end{pmatrix}.
\end{align*}

The relation $a_1^*B = \mu Ba_1^*$ takes the form:

\begin{align*}
b_{n+1}^*T_n &= \mu T_n b_n^*.
\end{align*}

If $x \neq 0$, then $b_k = \mu^{k-1}b_1$, and from $a_1^*a_1 = 1 - a_1a_1^*$ we have:

\begin{align*}
b_1^*b_1 &= 1 - b_1b_1^*, \\
\mu^2b_1b_1^* &= 1 - \mu^2b_1b_1^*.
\end{align*}

The latter equations are compatible only in the case $\mu^2 = 1$. This implies that $x = 0$, $B = 0$, $A = 0$.

Let us denote $B_k = a_ka_1 + \mu a_1a_k$, $k > 2$; then from the basic relations we have:

\begin{align*}
B_k^*B_k &= \mu^2B_kB_k^* + \mu^2(1 - \mu^2) \sum_{1 < i < k} B_iB_i^* + (1 + \mu^2)(1 - \mu^4)AA^*.
\end{align*}
It follows from the boundedness of $B_k$ that applying induction we obtain $B_k = 0$, $k = 3, \ldots, d$. Hence we have $a_k a_1 + \mu a_1 a_k = 0$, and in the irreducible representation

$$a_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes 1$$

$$a_k = \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix} \otimes \hat{a}_k, \quad k > 1.$$  

where $\{\hat{a}_k, k > 1\}$ satisfy $\mu$-CAR with $d - 1$ generators. It is obvious that $a_j^2 = 0$ if and only if $\hat{a}_j^2 = 0$ and $a_j a_i + \mu a_i a_j = 0$ if and only if its relation holds for $\hat{a}_j$, $\hat{a}_i$. Then the induction on $d$ completes the proof.

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References


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