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# BAND-SUMS ARE RIBBON CONCORDANT TO THE CONNECTED SUM

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ABSTRACT. We show that an arbitrary band-connected sum of two or more knots are ribbon concordant to the connected sum of these knots. As an application we consider which knot can be a nontrivial band-connected sum.

## 1. INTRODUCTION

In this paper we consider oriented knots and links in the oriented 3-sphere. Let $L$ be a split link with $n$ components $K_1, \dots, K_n$. Connect the components of $L$ via $n-1$ bands $b_1, \dots, b_{n-1}$ such that the orientation of the bands is consistent with that of $L$ (Figure 1). The knot obtained from $L$ by surgeries along these bands is called the *band-connected sum of* $K_1, \dots, K_n$ *along* $b_1, \dots, b_{n-1}$; refer to [5] for a more detailed definition.

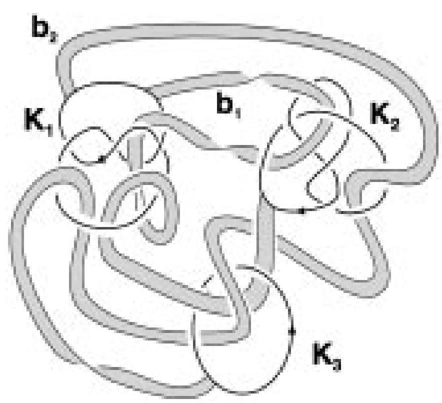


FIGURE 1

The bands and the band-connected sum are *trivial* if there are disjoint $n-1$ spheres $S_1, \dots, S_{n-1}$ in $S^3 - \bigcup_{i=1}^{n} K_i$ such that a core of $b_i$ intersects $S_i$ transversely in a single point but is disjoint from $S_j$ $(j \neq i)$. A trivial band-connected sum of $K_1, \dots, K_n$ is the connected sum $K_1 \# \cdots \# K_n$.

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A band-connected sum depends on the choice of bands. However, we show that the knot concordance class of a band-connected sum is uniquely determined by the split link $L$. In fact, a stronger statement is proved. To state the main theorem (Theorem 1.1) we need the notion of ribbon concordance introduced by C. Gordon [3].

**Definition.** Let $K_0$ and $K_1$ be knots in $S^3$. $K_1$ is *ribbon concordant to* $K_0$ (and write $K_1 \geq K_0$) if there is a concordance $C$ in $S^3 \times I$ between $K_1 \subset S^3 \times \{1\}$ and $K_0 \subset S^3 \times \{0\}$ such that the restriction to $C$ of the projection $S^3 \times I \to I$ is a Morse function with no local maxima. $C$ is a *ribbon concordance from* $K_1$ *to* $K_0$.

**Theorem 1.1.** *Any band-connected sum of knots* $K_1, \ldots, K_n$ *is ribbon concordant to the connected sum of* $K_1, \ldots, K_n$.

If a band-connected sum $K$ of $K_1, \ldots, K_n$ is minimal with respect to $\geq$, then $K \cong K_1 \# \cdots \# K_n$ by Theorem 1.1. Sufficient conditions for a knot to be minimal are studied by Gordon [3], Miyazaki [9]. The set of minimal knots includes (possibly trivial) torus knots [3] and iterated torus knots [9]. Hence we obtain the following result first proved by Howie and Short [5].

**Corollary 1.2.** *If a band-connected sum of* $K_1, \ldots, K_n$ *is a trivial knot, then all* $K_i$ *are trivial.*

A nontrivial band-connected sum of more than two knots can be unknotted [5, Figure 2], but Scharlemann [11] proves that if a band-connected sum of two knots is unknotted, then the band is trivial. So let us restrict our attention to band-connected sums of two knots. We write $K_1 \#_b K_2$ for the band-connected sum of $K_1$ and $K_2$ along a band $b$. A knot $K$ is *band-prime* if $K$ is a prime knot and cannot be expressed as a nontrivial band-connected sum of two knots. We shall prove:

**Theorem 1.3.** *If* $K_1 \#_b K_2$ *is a minimal knot with respect to* $\geq$, *then* $b$ *is a trivial band. In particular, a prime, minimal knot* (*e.g., a torus knot*) *is band-prime.*

*Remark.* Other sufficient conditions on band-primeness are obtained by T. Kobayashi [7].

As another application of Theorem 1.1 we consider when a band-connected sum of two knots can be a fibered knot. Developing Rapaport's ideas in [10], Silver [14] defined an invariant "band spread" for a ribbon concordance; he showed that if $K_1$ is a fibered knot and the band spread for $K_1 \geq K_0$ satisfies some equality, then $K_0$ is also fibered. We shall observe that a "generalized band spread" permits Silver's argument to extend. The band spread of a ribbon concordance from $K_1 \#_b K_2$ to $K_1 \# K_2$ may not satisfy the condition in [14], but the generalized one does. Thus we obtain the following result.

**Theorem 1.4.** *If a band-connected sum of* $K_1$ *and* $K_2$ *is fibered, then both* $K_i$ *are fibered.*

*Remark.* Kobayashi [6] proved this result for a "band-connected sum of two links" by using pre-fiber surfaces and the theory of sutured manifolds.

We close with two questions. Rapaport's conjecture on knot-like groups [10] implies the affirmative answer to Question (2); see Remark (1) in §2 and [13], [14].

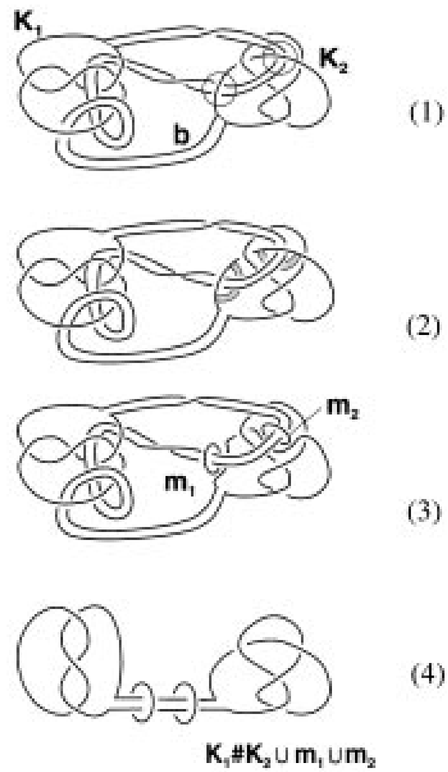


FIGURE 2

*Questions.* (1) If a band-connected sum of $K_1, \dots, K_n$ is fibered, then are all $K_i$ fibered?

(2) More generally, assume that $K_1$ is fibered and $K_1 \geq K_0$. Then is $K_0$ also fibered?

## 2. PROOFS

First note that if a ribbon concordance from $K_1$ to $K_0$ has $n$ local minima, then $K_1$ is a band-connected sum of the split link consisting of $K_0$ and $n$ trivial knots.

*Proof of Theorem 1.1.* We prove the theorem using pictures. Let $K_1, K_2$ and $b$ be as described in Figure 2(1). We change those crossings of the band $b$ and $K_2$ which are encircled in Figure 2(1) to get $b$ unlinked with $K_2$. Attach two bands to $K_2$ as in Figure 2(2). After surgeries along these bands, $b$ becomes free from $K_2$ and we have two "meridians" $m_1, m_2$ of $b$ (Figure 2(3)). Since $b$ is now a trivial band, after an isotopy we obtain a split link consisting of $K_1 \# K_2$ and two unknotted circles $m_1, m_2$ (Figure 2(4)). This process shows that $K_1 \#_b K_2$ is a band-connected sum of $K_1 \# K_2 \cup m_1 \cup m_2$ along two bands, so $K_1 \#_b K_2 \geq K_1 \# K_2$. The same arguments apply to the general case. □

*Proof of Theorem 1.3.* Let $K = K_1 \#_b K_2$ be minimal with respect to $\geq$. By Theorem 1.1 $K_1 \#_b K_2 \cong K_1 \# K_2$, so that $\text{genus}(K_1 \#_b K_2) = \text{genus}(K_1) + \text{genus}(K_2)$. This equality implies that there are disjoint Seifert surfaces $S_i$ for $K_i$ such that genus($K_i$) = genus($S_i$) and $S_1 \cup b \cup S_2$ is a Seifert surface for $K$ (Scharlemann [12], Gabai [2]). Assume for a contradiction that $b$ is nontrivial. Then, the band-connected sum $K$ with such a Seifert surface is a prime knot by Eudave-Muñoz [1]. It follows that $K_1$ or $K_2$, say $K_1$, is a trivial knot, hence $S_1$ is a disk. We see that the band $b$ is trivial, a contradiction. □

In the rest of this section we prove Theorem 1.4. Let $C \subset S^3 \times I$ be a ribbon concordance from $K_1$ to $K_0$ such that $C$ has $n$ local minima. Then $K_1$ is a band-connected sum of $K_0$ together with a trivial link with $n$ components. Let $G = \pi_1(S^3 \times I - C)$ and $G_i = \pi_1(S^3 \times \{i\} - K_i)$. Then $G$ has a presentation

$$G \cong \langle G_0, y_1, \dots, y_m | r_1, \dots, r_m \rangle. \tag{1}$$

The concordance group $G$ always has a presentation (1) in which $m = n$, $y_i$ represent meridians of the trivial link and $r_i$ correspond to the bands. Silver [14] defines a *band spread* $M_C$ for a presentation (1) of this type. This is motivated by Rapaport [10], who introduced the notion of "spread" to induce a Freiheitssatz for many-relator presentations. Since a band spread depends on the position of $C$, he requires that $M_C$ be least possible among all ribbon concordances isotopic to $C$. However, the definition of a band spread easily generalizes for every presentation (1). (The only required change is to use the substitution $y_i = x^s \bar{a}_i$ on [14, p. 101] instead of $y_i = x\bar{a}_i$, where $s = [y_i] \in G/G' \cong Z$.) We thus redefine $M_C$ to be the least possible (generalized) band spread among all presentations (1) of $G$. All arguments in [14] work for our $M_C$. It follows from Lemma 1, Theorem 2 and Corollary 1 of [14] that:

**Proposition 2.1.** *Suppose that $K_1$ is fibered. If there is a ribbon concordance $C$ from $K_1$ to $K_0$ such that $\pi_1(S^3 \times I - C)$ has a presentation* (1) *with $m = 1$, then $K_0$ is also fibered.*

*Remarks.* (1) Rapaport [10] conjectured: if $G$ is a knot-like group (i.e., the deficiency of $G$ is 1, and the abelianization $G/G'$ is infinite cyclic) and $G'$ is finitely generated, then $G'$ is free. This conjecture implies Theorem 1.4 and moreover the affirmative answer to Question (2) in the Introduction; see [14] for details. The above proof [14] of Proposition 2.1, in fact, proves Rapaport's conjecture for any ribbon concordance group which has a presentation (1) with $m = 1$.

(2) J. Hillman's recent result [4] shows that Rapaport's conjecture holds if the commutator subgroup $G'$ is almost finitely presented.

By Proposition 2.1, to prove Theorem 1.4 it suffices to show the lemma below.

**Lemma 2.2.** *Let $K_1 \#_b K_2$ be an arbitrary band-connected sum. Then there is a ribbon concordance $C \subset S^3 \times I$ from $K_1 \#_b K_2$ to $K_1 \# K_2$ such that $Y = S^3 \times I - \mathrm{int}N(C)$ has a handle decomposition*

$$Y = X_0 \times I \cup h^1 \cup h^2, \text{ and dually } Y = X_1 \times I \cup h^2 \cup h^3,$$

*where $X_0 = S^3 - \mathrm{int}N(K_1 \# K_2)$, $X_1 = S^3 - \mathrm{int}N(K_1 \#_b K_2)$, and $h^i$ is an $i$-handle.*

In general, if a ribbon concordance has $n$ local minima, then its exterior has a handle decomposition with $n$ 1-handles and $n$ 2-handles. However, it is not clear whether there exists a ribbon concordance from $K_1 \#_b K_2$ to $K_1 \# K_2$ with a single local minimum. The following idea of finding a handle decomposition is based on Marumoto [8, Corollary 1.10.1] and Thompson [15, Fig.2].

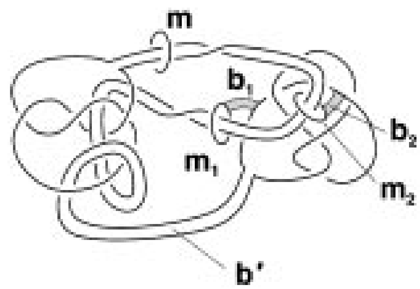


FIGURE 3

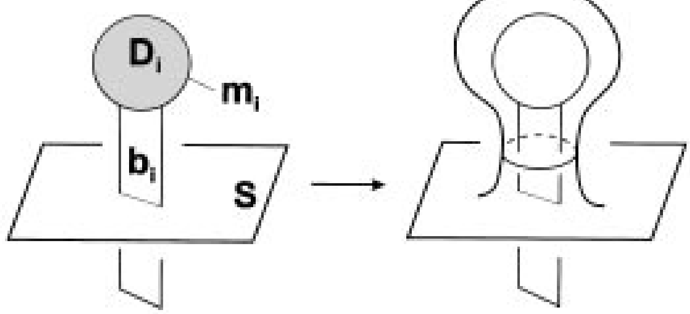


FIGURE 4

*Proof of Lemma 2.2.* We prove this for $K_1\#_b K_2$ described in Figure 2. The same arguments apply to the general case. Let $b'$ be the trivial band connecting $K_1$ and $K_2$ in Figures 2(3), 3. For simplicity, set $K = K_1\#_{b'}K_2 \cong K_1\#K_2$. As shown in the proof of Theorem 1.1, $K_1\#_b K_2$ is the band-connected sum of $K \cup m_1 \cup m_2$ along the bands $b_1, b_2$ (Figures 2(2), 3).

Let $m$ be a "meridian" of the band $b'$. Attach a 2-handle to $S^3 \times I$ along $m \subset S^3 \times \{0\}$ with 0-framing. Set $M = \partial(S^3 \times I \cup h^2) - S^3 \times \{1\}$; then $M \cong S^1 \times S^2$. Since $m_i$ are parallel to $m$, $m_i$ bound disjoint disks $D_i$ $(i = 1, 2)$ in $M$ such that $\text{int}D_i \cap (K \cup b_1 \cup b_2) = \emptyset$. On the other hand, $m$ bounds a disk in $S^3 \times \{0\}$ which is disjoint from $K \cup m_1 \cup m_2$ because $m$ is separated from $K$. By taking the union of this disk and a core of $h^2$, $M$ contains an essential 2-sphere $S$ such that $S \cap (K \cup m_1 \cup m_2) = \emptyset$. We may assume that the intersection of $S$ and $b_i$ consists of (possibly empty) arcs each of which meets the centerline of $b_i$ in a single point. If the intersection is empty, set $S' = S$. Otherwise, isotop $S$ along $b_i$ and pass over $D_i$ (and $h^2$). See Figure 4.

We then obtain an essential 2-sphere, $S'$, in $M$ which is disjoint from $K \cup \bigcup_{i=1}^{2}(b_i \cup D_i)$. Attach a 3-handle to $S^3 \times I \cup h^2$ along $S'$, and set $W = S^3 \times I \cup h^2 \cup h^3$. In the 3-sphere $\partial W - S^3 \times \{0\}$, the simple loop $K$ has the knot type of $K_1\#K_2$.

Now let $C = (K_1\#_b K_2) \times I \subset S^3 \times I$. Note that $\partial C$ is disjoint from the attaching spheres of $h^2$ and $h^3$, so $C$ is properly embedded in $W \cong S^3 \times I$. Let

$C' = C \cup \bigcup_{i=1}^{2} (b_i \cup D_i)$. Then $(C', \partial C')$ is isotopic to $(C, \partial C)$ in $(W, \partial W)$. Identify $W$ with $S^3 \times I$. We see that $\partial C' \cap S^3 \times \{1\} = K_1 \#_b K_2$ and $\partial C' \cap S^3 \times \{0\} = K$. Moreover, after an isotopy, $C' \subset S^3 \times I$ has no local maxima, two local minima corresponding to $D_1, D_2$, and two saddle points corresponding to $b_1, b_2$. Since $W - \mathrm{int}N(C') \cong (S^3 - \mathrm{int}N(K_1 \#_b K_2)) \times I \cup h^2 \cup h^3$, $C'$ is the desired ribbon concordance. □

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