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THE FUNCTION $(b^x - a^x)/x$: INEQUALITIES AND PROPERTIES

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ABSTRACT. In the article, some properties and inequalities of the function $\int_a^b t^{x-1}\mathrm{d}t$ are given by analytic method and the mathematical induction.

1. INTRODUCTION

Let

$$g(x) = (b^x - a^x)/x, \quad b > a > 0, \quad x \neq 0; \qquad (1)$$
$$g(0) = \ln b - \ln a.$$

Define a function $U_n(x;t)$ such that

$$U_0(x;t) = t^x, \quad x\partial U_n(x;t)/\partial x - (n+1)U_n(x;t) = U_{n+1}(x;t) \qquad (2)$$

for $n \in \mathrm{N}, t \in [a,b]$.

It is easy to see that

$$g^{(n)}(x) = \int_a^b (\ln t)^n t^{x-1}\mathrm{d}t, \quad b > a > 0, \quad n \in \mathrm{N}. \qquad (3)$$

In this article, by analytic method and mathematical induction, the functions $g(x)$ and $U_n(x,t)$ are researched, and some properties and inequalities of them are given, that is,

Proposition 1. *Let $g = g(x) = \int_a^b t^{x-1}dt$. Then, for $k, i, j \in \mathrm{N}$*

$$g^{(2(i+k)+1)}g^{(2(j+k)+1)} < g^{(2k)}g^{(2(i+j+k+1))}. \qquad (4)$$

The ratio $g^{(2(j+k)+1)}(x)/g^{(2k)}(x)$ is increasing in x.

Proposition 2. *The function $g(x)$ satisfies*

$$g^{(n)}(x) = (U_n(x;b) - U_n(x;a))/x^{n+1}, \qquad (5)$$
$$\partial U_n(x,t)/\partial t = x^{n+1}(\ln t)^n t^{x-1}. \qquad (6)$$

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Proposition 3. *The function $g(x)$ is absolutely and regularly monotonic on $(-\infty, +\infty)$ for $a > 1$, or on $(0, +\infty)$ for $b > a^{-1} > 1$, completely and regularly monotonic on $(-\infty, +\infty)$ for $0 < a < b < 1$, or on $(-\infty, 0)$ for $1 < b < a^{-1}$. Furthermore, $g(x)$ is absolutely convex on $(-\infty, +\infty)$.*

Proposition 4. *The function $g(x+\gamma)/g(x)$ is increasing (or decreasing) in x for $\gamma > 0$ (or $\gamma < 0$). $[g(x+t)/g(x)]^{1/t}, t \neq 0$, is increasing with t.*

Proposition 5. *For $\gamma \geq 1, x \geq 1, 0 < a < b$, the following inequality holds:*

$$\frac{b^{x+\gamma} - a^{x+\gamma}}{b^x - a^x} \geq \frac{x+\gamma}{x} \left(\frac{a+b}{2} \right)^{\gamma}. \tag{7}$$

But (7) may not hold for $0 < \gamma < 1$ or $0 < x < 1$. If $\gamma > 0, x > 0$, then

$$\frac{b^{x+\gamma} - a^{x+\gamma}}{b^x - a^x} \geq \frac{x+\gamma}{x} [ab]^{\gamma/2}. \tag{8}$$

Note that Proposition 5 refines and extends the inequalities in [2], and verifies the conjecture by the first author in [2].

Using the method of this article, we can generalize the extended means to a two-parameter family of nonhomogeneous mean values; see [3].

2. Definitions and lemmas

The following definitions and lemmas are necessary.

Definition 1. A function $f(t)$ is said to be *absolutely monotonic on* (a, b) if it has derivatives of all orders and $f^{(k)}(t) \geq 0, t \in (a, b), k \in \mathrm{N}$.

Definition 2. A function $f(t)$ is said to be *completely monotonic on* (a, b) if it has derivatives of all orders and $(-1)^k f^{(k)}(t) \geq 0, t \in (a, b), k \in \mathrm{N}$.

Definition 3. A function $f(t)$ is said to be *absolutely convex on* (a, b) if it has derivatives of all orders and $f^{(2k)}(t) \geq 0, t \in (a, b), k \in \mathrm{N}$.

Definition 4. A function $f(t)$ is said to be *regularly monotonic* if it and its derivatives of all orders have constant sign ($+$ or $-$; not all the same) on (a, b).

Lemma 1. *Let $f, h : [a, b] \to \mathrm{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a, b] \to \mathrm{R}^+$ be an integrable function. Then*

$$\int_a^b p(t)f(t)dt \int_a^b p(t)h(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)h(t)dt. \tag{9}$$

If one of the functions of f or h is nonincreasing and the other nondecreasing, then the inequality in (9) is reversed.

Lemma 2. *Let $f : [a, b] \to \mathrm{R}$ be a convex function; then*

$$f\left(\frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{10}$$

Inequalities (10) and (9) are called the Hermite-Hadamard and the Tchebycheff integral inequality, respectively.

Lemma 3. *For $x, y > 0, x \neq y$,*

$$\frac{x+y}{2} > \frac{x-y}{\ln x - \ln y} > \sqrt{xy}. \tag{11}$$

Inequality (11) is called the logarithmic mean inequality.
For proofs of these lemmas, see [1, 4].

3. Proofs of the propositions

3.1. Inequality (4) is a special case of Lemma 1 applied to the functions $p(t) = (\ln t)^{2k} t^{x-1}, f(t) = (\ln t)^{2i+1}$ and $h(t) = (\ln t)^{2j+1}$ for $i, j, k \in \mathrm{N}, x \in \mathrm{R}$ and $t \in [a, b]$.

Inequality (4) and direct calculation produces

$$\left(\frac{g^{(2(j+k)+1)}}{g^{(2k)}}\right)' = \frac{g^{(2(j+k+1))} g^{(2k)} - g^{(m)} g^{(2(j+k+1))}}{\left(g^{(2k)}\right)^2} > 0.$$

Therefore, the desired Proposition 1 follows.

3.2. Using (2) and direct computation yield

$$\begin{aligned}
g^{(n+1)}(x) &= \frac{d}{dx}\left(g^{(n)}(x)\right) = \frac{d}{dx}\left(\frac{U_n(x;b) - U_n(x;a)}{x^{n+1}}\right) \\
&= \frac{[x\partial U_n(x,b)/\partial x - (n+1)U_n(x,b)] - [x\partial U_n(x,a)/\partial x - (n+1)U_n(x,a)]}{x^{n+2}} \\
&= \frac{U_{n+1}(x;b) - U_{n+1}(x;a)}{x^{n+2}}.
\end{aligned}$$

By mathematical induction on n, (5) is valid.

Differentiating (2) with respect to t gives

$$\begin{aligned}
\partial U_{n+1}/\partial t =& x\partial^2 U_n/\partial x \partial t - (n+1)\partial U_n/\partial t \\
=& x\partial(\partial U_n/\partial t)/\partial x - (n+1)\partial U_n/\partial t.
\end{aligned}$$

Therefore, from mathematical induction on n, we obtain (6).

This completes the proof of Proposition 2.

3.3. First, we consider the case of $x > 0$. It is clear that $g^{(2k)}(x) \geq 0$, $g(x)$ is an absolutely convex function on $(0, +\infty)$.

When $a > 1, \partial U_n/\partial t > 0$, thus $U_n(x, t)$ increases with respect to t, and $g^{(n)}(x) > 0$ follows from (5). Therefore $g(x)$ is absolutely monotonic on $(0, +\infty)$.

When $0 < a < b < 1, \partial U_{2k+1}/\partial t < 0$; then $U_{2k+1}(x, t)$ decreases in t, thus $g^{(2k+1)}(x) < 0$, that is, $(-1)^k g^{(k)}(x) > 0, g(x)$ is completely monotonic on $(0, +\infty)$.

Direct computation results in

$$\lim_{x \to 0^+} g^{(2k+1)}(x) = [(\ln b)^{2(k+1)} - (\ln a)^{2(k+1)}]/2(k+1) > 0 \tag{12}$$

for $b > a^{-1} \geq 1$. Since $g^{(2k+1)}(x)$ is increasing, $g^{(2k+1)}(x) > 0$ for $x > 0$. Therefore, $g(x)$ is absolutely monotonic on $(0, +\infty)$ for $b > a^{-1} \geq 1$.

Second, we consider the case of $x < 0$. For $n = 2k + 1, k \in \mathrm{N}$, when $a > 1$, $\partial U_n/\partial t > 0, U_n(x, t)$ increases in t, from (5) it follows that $g^{(2k+1)}(x) > 0$. When $0 < a < b < 1, \partial U_n/\partial t < 0, U_n(x, t)$ is decreasing with t, $g^{(2k+1)}(x) < 0$.

For $n = 2k, \partial U_n/\partial t \leq 0, U_n(x, t)$ is decreasing in t, $g^{(2k)}(x) \geq 0$, thus $g(x)$ is absolutely convex on $(-\infty, 0)$.

Therefore, $g(x)$ is an absolutely monotonic function on $(-\infty, 0)$ for $a > 1$. $g(x)$ is completely monotonic on $(-\infty, 0)$ for $0 < a < b < 1$.

Since $g^{(2k+1)}(x), x < 0$, is increasing, inequality (12) is reversed for $1 < b < a^{-1}$, thus $g^{(2k+1)}(x) < 0$ for $1 < b < a^{-1}$ and $x < 0$, $g(x)$ is a completely monotonic function for $1 < b < a^{-1}$ on $(-\infty, 0)$.

From the definitions in section 2, Proposition 3 is valid.

3.4. Let $f(x) = g(x+\gamma)/g(x), \gamma \neq 0$. From Proposition 1, it is clear that $g'(x)/g(x)$ is increasing, thus

$$g'(x+\gamma)/g(x+\gamma) \geq g'(x)/g(x) \tag{13}$$

holds for $\gamma > 0$, and is reversed for $\gamma < 0$. Straightforward computation leads to

$$f'(x) = [g'(x+\gamma)g(x) - g(x+\gamma)g'(x)]/g^2(x). \tag{14}$$

Combining (13) and (14) produces that $f'(x) > 0$ for $\gamma > 0$ and $f'(x) < 0$ for $\gamma < 0$, therefore $f(x)$ increases (or decreases) for $\gamma > 0$ (or < 0).

Assume

$$p(t,\theta) = [g(t+\theta)/g(t)]^{1/\theta}, \theta \neq 0;$$
$$p(t,0) = \exp(g'(t)/g(t)), t \in \mathrm{R}.$$

It is clear that $p(t,0)$ increases with t. Computating straightforwardly gives

$$[\ln p(t,\theta)]'_\theta = \left[\frac{g'(t+\theta)}{g(t+\theta)}\theta - \ln\frac{g(t+\theta)}{g(t)}\right]/\theta^2.$$

By the mean value theorem, it is easy to see that

$$\ln\frac{g(t+\theta)}{g(t)} = \frac{g'(t+\xi)}{g(t+\xi)}\theta < \frac{g'(t+\theta)}{g(t+\theta)}\theta,$$

where ξ is between 0 and $\theta, \theta \neq 0$. Therefore $[\ln p(t,\theta)]'_\theta > 0$, and $p(t,\theta)$ is increasing in $\theta, \theta \neq 0$. The proof of Proposition 4 is completed.

3.5. Since $f(x)$ is increasing for $\gamma > 0$

$$\frac{x(b^{x+\gamma} - a^{x+\gamma})}{(x+\gamma)(b^x - a^x)} \geq \frac{b^{1+\gamma} - a^{1+\gamma}}{(1+\gamma)(b-a)}, \quad x \geq 1, \gamma > 0, 0 < a < b. \tag{15}$$

Since $t^\gamma (\gamma \geq 1)$ is convex, from (10) we have

$$\left(\frac{a+b}{2}\right)^\gamma \leq \frac{1}{b-a}\int_a^b t^\gamma \mathrm{d}t = \frac{b^{1+\gamma} - a^{1+\gamma}}{(b-a)(1+\gamma)}, \gamma \geq 1. \tag{16}$$

Combining (15) and (16) yields (7).

Since t^γ $(0 < \gamma < 1)$ is concave, then (16) is reversed without equality. Hence (7) may not hold for $0 < \gamma < 1$.

Since $f(x)$ increases for $\gamma > 0, x > 0$, then we have

$$\frac{x(b^{x+\gamma} - a^{x+\gamma})}{(x+\gamma)(b^x - a^x)} \geq \frac{b^\gamma - a^\gamma}{\gamma(\ln b - \ln a)}, \quad x \geq 0, \gamma > 0, \tag{17}$$

which, combined with the logarithmic mean inequality (11), yields (8), but (7) may not hold. Therefore, Proposition 5 is verified.

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