PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 11, November 1998, Pages 3417–3425
S 0002-9939(98)04447-5

PRODUCTS OF CONSTANT CURVATURE SPACES WITH A BROWNIAN INDEPENDENCE PROPERTY

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(Communicated by Stanley Sawyer)

Abstract. The time and place Brownian motion on the product of constant curvature spaces first exits a normal ball of radius ϵ centered at the starting point of the Brownian motion are considered. The asymptotic expansions, as ϵ decreases to zero, for joint moments of the first exit time and place random variables are computed with error $O(\epsilon^{10})$. It is shown that the first exit time and place are independent random variables only if each factor space is locally flat or of dimension three.

1. Introduction

It is well known that, for Brownian motion started at the origin in $\mathbb{R}^n$, the exit time and place from a ball centered at the origin are independent random variables. Among Riemannian manifolds, this property also holds for locally harmonic spaces [KO1], [L2]. In [H1], it is shown that the product of constant curvature spaces $S^3 \times H^3$, which is not locally harmonic, also satisfies this independence property. In this paper, we completely determine which products of constant curvature spaces have this independence property.

Let M be an n-dimensional Riemannian manifold. Let $B_m(\epsilon)$ denote the geodesic ball of radius ϵ about m in M. Let $S_m(\epsilon)$ denote the boundary of $B_m(\epsilon)$. Let (X_t) be Brownian motion on $B_m(\epsilon)$, started from m. The infinitesimal generator of (X_t) is $\frac{1}{2}\Delta$, where Δ is the Laplace-Beltrami operator on functions. Let $T(\epsilon)$ be the first exit time of X_t from $B_m(\epsilon)$:

$$T(\epsilon) = \inf\{t > 0 : d(m, X_t) \geq \epsilon\}. \tag{1.1}$$

Then $X_{T(\epsilon)}$ is the first exit place of X_t from $B_m(\epsilon)$.

In this notation, we have the result from [H1] mentioned above.

Theorem 1.1. *If $M = S^3 \times H^3$, where S^3 and H^3 are a three-dimensional sphere and hyperbolic space, respectively, and $B_m(\epsilon)$ is a normal ball, then $T(\epsilon)$ and $X_{T(\epsilon)}$ are independent random variables.*

As remarked in [H1], this result is easily extended to any M that is a product of spaces of constant sectional curvature where each factor is of dimension three or is locally flat (sectional curvature is zero or the dimension is one).

Received by the editors February 3, 1997 and, in revised form, March 27, 1997.
1991 *Mathematics Subject Classification.* Primary 58G32; Secondary 53B20, 60J65.
Key words and phrases. Brownian motion, symmetric space, exit time, exit place.

We fix a point $m \in M$ and represent all tensors with respect to normal coordinates at m. No distinction is made between upper and lower indices and we use the summation convention for any repeated indices, unless otherwise noted. Let R_{ijkl} denote the Riemann curvature tensor, $\rho_{ij} = R_{iaja}$ denote the Ricci curvature tensor, and $\tau = \rho_{aa}$ denote scalar curvature. The metric tensor is denoted g_{ij} and equals the Kronecker delta δ_{ij} at m. In [KO1] and [H2] it is shown that if $T(\epsilon)$ and $X_{T(\epsilon)}$ are independent for all normal balls at each $m \in M$ then M has constant scalar curvature,

(1.2)
$$(3\nabla_p\nabla_p\rho_{ij} - 4\rho_{ip}\rho_{jp} + 2R_{ipjq}\rho_{pq} + 2R_{ipqr}R_{jpqr})_m = (1/n)[(2\|R\|^2 - 2\|\rho\|^2)g_{ij}]_m$$

holds for every $m \in M$, and $\|R\|^2 - \|\rho\|^2$ is constant on M.

It can be shown, for example, that these conditions are satisfied by any manifold $M = M^{n_1}(\kappa_1) \times M^{n_2}(\kappa_2)$ where $M^{n_i}(\kappa_i)$ is an n_i dimensional manifold with constant sectional curvature κ_i and where $n_1 = n_2$ and $|\kappa_1| = |\kappa_2|$ [KO2]. However, by considering higher order asymptotics, we will show that $T(\epsilon)$ and $X_{T(\epsilon)}$ are independent for all normal balls centered at $m \in M$ only if $n_i = 1$ or $n_i = 3$ or $\kappa_i = 0$, $i = 1, 2$. In fact, we show this result for the product of K constant curvature spaces. Thus, further information is obtained from higher order asymptotics and we conjecture that these higher order asymptotics could be used to classify more general symmetric spaces with independent first exit time and place from normal balls.

2. Perturbation method and reduction process

We use the extension of the perturbation method of [GP] and [L1] described in [H2]. We will translate the problem from a normal ball about $m \in M$ to the unit ball in T_mM, the tangent space at m. We consider values of ϵ less than ϵ_1 where ϵ_1 is chosen such that the exponential map is non-singular in $B_0(\epsilon_1)$, the ball of radius ϵ_1 in T_mM. Let B and S denote the unit ball and unit sphere, respectively, in T_mM. We define $\Phi_\epsilon : C^\infty(\overline{B}) \to C^\infty(\overline{B_m(\epsilon)})$ by

$$(\Phi_\epsilon f)(\exp_m x) \doteq f(x/\epsilon). \tag{2.1}$$

Working in the unit ball, we find it convenient to use homogeneous harmonic polynomials. Let $\sum_{P(i_1\dots i_k)}$ denote summation over the permutations of the indices $i_1, \dots, i_k$. It follows that we can define $Q_{i_1\dots i_k}(z)$, a homogeneous harmonic polynomial of degree k, by the following [VK, Section 9.2.3]:

(2.2)
$$z_{i_1}\cdots z_{i_k} + \sum_{j=1}^{[k/2]} \sum_{P(i_1\dots i_k)} \frac{(-1)^j\,|z|^{2j}\,\delta_{i_1 i_2}\cdots\delta_{i_{2j-1}i_{2j}}\,z_{i_{2j+1}}\cdots z_{i_k}}{2^j\,j!(k-2j)!(n+2k-4)(n+2k-6)\cdots(n+2k-2j-2)}.$$

In particular, we will use the following.

(2.3)
$$Q_{ij}(z) = z_iz_j - \frac{1}{n}\delta_{ij}|z|^2,$$
$$Q_{ijkl}(z) = z_iz_jz_kz_l - \frac{1}{4(n+4)}\sum_{P(ijkl)}\delta_{ij}z_kz_l|z|^2 + \frac{1}{8(n+2)(n+4)}\sum_{P(ijkl)}\delta_{ij}\delta_{kl}|z|^4.$$

For a Riemannian manifold M, the infinitesimal generator of Brownian motion is $A = \frac{1}{2}\Delta$ which has the following homogeneous decomposition with respect to normal coordinates (x^j) [GP]:

$$A = \frac{1}{2}\Delta_{-2} + \frac{1}{2}\sum_{i=0}^{\infty}\Delta_i \qquad (\Delta_{-1} = 0). \tag{2.4}$$

Here Δ_{-2} is the ordinary Euclidean Laplacian and Δ_i is a second order linear differential operator with polynomial coefficients of the form $\Delta_i = q_i^j\partial_j + q_i^{jk}\partial_j\partial_k$ where q_i^j and q_i^{jk} are respectively $(i+1)$-degree and $(i+2)$-degree homogeneous polynomials in (x^j). Δ_i maps k-degree homogeneous polynomials to $(k+i)$-degree homogeneous polynomials.

From [H2] we have the following result.

Theorem 2.1. *Let f be a smooth function on S. Suppose that u_j and v_j, $0 \le j \le k$, satisfy*

$$\begin{gathered}\Delta_{-2}u_0 = 0 \text{ in } B, \quad u_0 = f \text{ on } S,\\ \Delta_{-2}u_j + \sum_{i=2}^{j}\Delta_{i-2}u_{j-i} = 0 \text{ in } B, \quad u_j = 0 \text{ on } S \quad (1 \le j \le k),\\ \Delta_{-2}v_j + \sum_{i=2}^{j}\Delta_{i-2}v_{j-i} + 2u_j = 0 \text{ in } B, \quad v_j = 0 \text{ on } S \quad (0 \le j \le k).\end{gathered} \tag{2.5}$$

Let $\hat{u}_\epsilon$ and $\hat{v}_\epsilon$ be defined as follows:

$$\hat{u}_\epsilon = \sum_{j=0}^{k}\epsilon^j u_j, \quad \hat{v}_\epsilon = \sum_{j=0}^{k}\epsilon^{j+2}v_j. \tag{2.6}$$

Then

$$\begin{aligned}E_m[(\Phi_\epsilon f)(X_{T(\epsilon)})] &= \hat{u}_\epsilon(0) + O(\epsilon^{k+1}),\\ E_m[T(\epsilon)(\Phi_\epsilon f)(X_{T(\epsilon)})] &= \hat{v}_\epsilon(0) + O(\epsilon^{k+3}).\end{aligned} \tag{2.7}$$

Note that (2.5) defines u_j and v_j recursively. Also, the sums in (2.5) are empty when $j = 1$ and it follows that u_1 and v_1 are identically zero.

Set $f \equiv 1$. Then $\hat{u}_\epsilon \equiv 1$ and Theorem 2.1 implies:

Corollary 2.2. *Let $f \equiv 1$. Suppose that u_j and v_j, $0 \le j \le k$, satisfy (2.5) and $\hat{u}_\epsilon$ and $\hat{v}_\epsilon$ are defined as in (2.6). Then $\hat{u}_\epsilon \equiv 1$ and*

$$E_m[T(\epsilon)] = \hat{v}_\epsilon(0) + O(\epsilon^{k+3}). \tag{2.8}$$

As described in [H2], the values $u_j(0)$, $1 \le j \le k$, and $v_j(0)$, $0 \le j \le k$, defined by solutions of (2.5) can be expressed in terms of u_0 and its derivatives evaluated at 0. In the remainder of this section we outline here the procedure which is proved in detail in [H2].

Define the differential operator A_k^j as follows:

$$A_k^j \doteq \begin{cases}\Delta_k, & j = 0,\\ \Delta_{-2}A_k^{j-1} - A_k^{j-1}\Delta_{-2}, & j \ge 1.\end{cases} \tag{2.9}$$

It follows easily by induction that each A_k^j has the following properties:

Lemma 2.3. *A_k^j is a linear differential operator, of order $2+j$, with polynomial coefficients of degree $\leq 2+k-j$ and $\geq 1+k-2j$ (where a polynomial of negative degree is understood to be the 0 polynomial). Furthermore,*

$$\Delta_{-2}^m \Delta_k = \sum_{j=0}^{m} \binom{m}{j} A_k^j \Delta_{-2}^{m-j}. \tag{2.10}$$

For the homogeneous harmonic polynomial of degree k defined by (2.2), define the coefficients $Q_{i_1\ldots i_k}^{j_1\ldots j_k}$ by the relation

$$Q_{i_1\ldots i_k}(z) = \sum_{j_1,\ldots,j_k=1}^{n} Q_{i_1\ldots i_k}^{j_1\ldots j_k} z_{j_1}\ldots z_{j_k}. \tag{2.11}$$

Without loss of generality, we may take $Q_{i_1\ldots i_k}^{j_1\ldots j_k}$ to be symmetric in the indices $j_1,\ldots,j_k$. Define the differential operator

$$Q_{i_1\ldots i_k}(\nabla) \doteq \sum_{j_1,\ldots,j_k=1}^{n} Q_{i_1\ldots i_k}^{j_1\ldots j_k} \partial_{j_1}\ldots\partial_{j_k}. \tag{2.12}$$

Define the following dimension constants for $k \geq 0$:

$$\phi_h^k \doteq \begin{cases} 1, & h = 0, \\ \dfrac{1}{2^h h!\,(n+2k)(n+2k+2)\cdots(n+2k+2h-2)}, & h > 0. \end{cases} \tag{2.13}$$

Using Pizetti's formula in $\mathbb{R}^n$ [CH, pp. 287–289], we have the following lemma.

Lemma 2.4. *Suppose $\Delta_{-2}^{m+1} u = 0$ in B and $u = 0$ on S. Then*

$$\begin{aligned} u(0) &= -\sum_{h=1}^{m} \phi_h^0 (\Delta_{-2}^h u)(0), \\ [Q_{i_1\ldots i_k}(\nabla)u](0) &= -\sum_{h=1}^{m} \phi_h^k [Q_{i_1\ldots i_k}(\nabla)\Delta_{-2}^h u](0). \end{aligned} \tag{2.14}$$

By successively applying (2.5), (2.10), and (2.14), the values of $u_j(0)$ and $v_j(0)$ are found in terms of u_0 and its derivatives evaluated at 0 as desired. These values can be expressed as follows (using the Poisson formula).

Lemma 2.5.

$$(\partial_{i_1}\partial_{i_2}\ldots\partial_{i_k} u_0)(0) = \int_S n(n+2)\cdots(n+2k-2) Q_{i_1 i_2\ldots i_k}(z) f(z)\, d\sigma(z) \tag{2.15}$$

where σ is the uniform probability measure on S.

When a harmonic homogeneous polynomial $Q_{p_1\ldots p_m}(z)$ is substituted for $f(z)$, we have an explicit computation for the integral.

Lemma 2.6.

$$\begin{aligned} &\int_S n(n+2)\cdots(n+2k-2) Q_{i_1\ldots i_k}(z) Q_{p_1\ldots p_m}(z)\, d\sigma(z) \\ &\qquad = \begin{cases} k!\, Q_{i_1\ldots i_k}^{p_1\ldots p_k}, & k = m, \\ 0, & k \neq m, \end{cases} \end{aligned} \tag{2.16}$$

where $Q_{i_1\ldots i_k}^{p_1\ldots p_k}$ are the coefficients of $Q_{i_1\ldots i_k}(z)$ given by (2.11).

3. Decomposition of the Laplacian for constant curvature spaces

Using the reduction procedure of the preceding section a straightforward (but extremely lengthy) calculation gives as many terms as desired in the expansion of

$$E_m[T(\epsilon)(\Phi_\epsilon f)(X_{T(\epsilon)})] - E_m[T(\epsilon)]E_m[(\Phi_\epsilon f)(X_{T(\epsilon)})] \tag{3.1}$$

in powers of ϵ. Note that if the first exit time and place are independent then this expression equals zero. In order to compute this expansion within $O(\epsilon^{10})$ it is necessary to compute $u_j(0)$, $1 \le j \le 7$, and $v_j(0)$, $0 \le j \le 7$. This requires the homogeneous decomposition of the Laplacian up to Δ_5. For a general Riemannian manifold M, this is a long and difficult calculation involving complicated curvature terms. However, if M is a manifold with constant sectional curvature, the operators Δ_j are somewhat simpler and the computation becomes feasible.

Proposition 3.1. *Let M be a Riemannian manifold of dimension n with constant sectional curvature κ. Then the homogeneous decomposition of the Laplacian with respect to normal coordinates has $\Delta_j = 0$ if j is odd and*

$$\begin{aligned}
\Delta_0 &= \frac{\kappa}{3}\{|x|^2\Delta_{-2} - x^i x^j \partial_i\partial_j - 2(n-1)x^i\partial_i\},\\
\Delta_2 &= \frac{\kappa^2}{45}\{3|x|^4\Delta_{-2} - 3|x|^2 x^i x^j \partial_i\partial_j - 4(n-1)|x|^2 x^i\partial_i\},\\
\Delta_4 &= \frac{\kappa^3}{945}\{10|x|^6\Delta_{-2} - 10|x|^4 x^i x^j \partial_i\partial_j - 12(n-1)|x|^4 x^i\partial_i\}.
\end{aligned} \tag{3.2}$$

Proof. We show this for the case $\kappa > 0$. Here M is locally isometric to a sphere $S^n(\kappa)$. The case with $\kappa < 0$ is similar with trigonometric functions replaced by hyperbolic functions. Let $\{x^i\}$ be normal coordinates at m and $r = |x|$ be the geodesic distance from m. Then the Laplacian for S^n is

$$\begin{aligned}
\Delta &= \frac{\partial^2}{\partial r^2} + (n-1)\sqrt{\kappa}\cot(r\sqrt{\kappa})\frac{\partial}{\partial r} + \frac{\kappa}{\sin^2(r\sqrt{\kappa})}\Delta_{S^{n-1}}\\
&= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}\\
&\quad + \frac{n-1}{r}\left(r\sqrt{\kappa}\cot(r\sqrt{\kappa}) - 1\right)\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\kappa r^2}{\sin^2(r\sqrt{\kappa})} - 1\right)\Delta_{S^{n-1}}
\end{aligned} \tag{3.3}$$

where $\Delta_{S^{n-1}}$ is the Laplacian on the $(n-1)$-dimensional unit sphere (the angular components). Now use the facts

$$\begin{aligned}
r\partial r &= x^i\partial_i,\\
\Delta_{-2} &= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}.
\end{aligned} \tag{3.4}$$

These imply

$$\Delta_{S^{n-1}} = r^2\Delta_{-2} - (n-1)x^i\partial_i - x^i x^j\partial_i\partial_j. \tag{3.5}$$

Define the functions

$$\begin{aligned}
\Psi_1(t^2) &\doteq \frac{1}{t^2}\left(\frac{t^2}{\sin^2 t} - 1\right),\\
\Psi_2(t^2) &\doteq \frac{1 - t\cot t}{t^2}.
\end{aligned} \tag{3.6}$$

Then we have

$$\begin{aligned}\Delta = \Delta_{-2} + \kappa r^2 \Psi_1(\kappa r^2)\Delta_{-2} - \kappa \Psi_1(\kappa r^2) x^i x^j \partial_i \partial_j \\ - (n-1)\kappa\left(\Psi_1(\kappa r^2) + \Psi_2(\kappa r^2)\right) x^i \partial_i.\end{aligned} \tag{3.7}$$

Finally, using the Maclaurin series for Ψ_1 and Ψ_2,

$$\begin{aligned}\Psi_1(t^2) &= \frac{1}{3} + \frac{1}{15}t^2 + \frac{2}{189}t^4 + \dots, \\ \Psi_2(t^2) &= \frac{1}{3} + \frac{1}{45}t^2 + \frac{2}{945}t^4 + \dots,\end{aligned} \tag{3.8}$$

we get the decomposition in (3.2). □

4. Results for products of constant curvature spaces

Now consider the metric product manifold $M = M_1 \times \dots \times M_K$ where, for $1 \le \alpha \le K$, M_α is a manifold of dimension n_α and has constant sectional curvature κ_α. Let $n = \sum_{\alpha=1}^{k} n_\alpha$ and $N_\beta = \sum_{\alpha=1}^{\beta} n_\alpha$. Consider the normal coordinate system $\{x^j\}_{1\le j\le n}$ where $\{x^j\}_{N_{\alpha-1}+1\le j\le N_\alpha}$ is a normal coordinate system for the αth factor space M_α. Let ${}_\alpha\Delta$ and ${}_\alpha\Delta_j$ be the Laplacian and its homogeneous decomposition, respectively, with respect to these coordinates on M_α. Then the Laplacian for M is

$$\Delta = \sum_{\alpha=1}^{K} {}_\alpha\Delta \tag{4.1}$$

and has homogeneous decomposition given by

$$\Delta_j = \sum_{\alpha=1}^{K} {}_\alpha\Delta_j. \tag{4.2}$$

Define the Kronecker delta for each factor by

$${}_\alpha\delta_{ij} = \begin{cases} 1, & \text{if } N_{\alpha-1}+1 \le i = j \le N_\alpha, \\ 0, & \text{otherwise.}\end{cases} \tag{4.3}$$

We use many iterations of the reduction process described in Section 2 with the decomposition of the Laplacian given above. The details of each step are too long to show here. The result of this process is the following.

Theorem 4.1. *Let M be given as above. Then*

$$\begin{aligned} E_m[T(\epsilon)(\Phi_\epsilon f)(X_{T(\epsilon)})] - E_m[T(\epsilon)]E_m[(\Phi_\epsilon f)(X_{T(\epsilon)})] \\ = \epsilon^6 \int_S B_{ij} Q_{ij}(z) f(z)\, d\sigma(z) + \epsilon^8 \int_S C_{ij} Q_{ij}(z) f(z)\, d\sigma(z) \\ + \epsilon^8 \int_S D_{ijkl} Q_{ijkl}(z) f(z)\, d\sigma(z) + O(\epsilon^{10}),\end{aligned} \tag{4.4}$$

where

$$B_{ij} = -\frac{\sum_{\alpha=1}^{K} \kappa_\alpha^2 (n_\alpha - 1)(n_\alpha - 3)\, {}_\alpha\delta_{ij}}{180(n+4)^2(n+6)} \tag{4.5}$$

and

$$
\begin{aligned}
C_{ij} = \Bigg(&6n(n+4)\sum_{\alpha=1}^{K}\kappa_\alpha^3(n_\alpha-1)(n_\alpha-3)(2n_\alpha-17)_\alpha\delta_{ij} \\
&-7n(n-16)\sum_{\alpha=1}^{K}\kappa_\alpha^3(n_\alpha-1)^2(n_\alpha-3)_\alpha\delta_{ij} \\
&-35(3n+8)\sum_{\alpha,\beta=1}^{K}\kappa_\alpha^2\kappa_\beta(n_\alpha-1)(n_\alpha-3)(n_\beta-1)(n_\beta)_\alpha\delta_{ij} \\
&-35(n+8)\sum_{\alpha,\beta=1}^{K}\kappa_\alpha\kappa_\beta^2(n_\alpha-1)(n_\beta-1)(n_\beta-3)(n_\beta)_\alpha\delta_{ij}\Bigg) \\
&\Big/ \big(18900n(n+4)^3(n+6)(n+8)\big)
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
D_{ijkl} = &-\frac{\sum_{\alpha=1}^{K}\kappa_\alpha^3(n_\alpha-1)(n_\alpha-3)({}_\alpha\delta_{ij})({}_\alpha\delta_{kl})}{1890(n+8)^2(n+10)} \\
&+\frac{\sum_{\alpha,\beta=1}^{K}\kappa_\alpha^2\kappa_\beta(n_\alpha-1)(n_\alpha-3)(n_\beta-1)({}_\alpha\delta_{ij})({}_\beta\delta_{kl})}{2160(n+4)^2(n+6)}.
\end{aligned}
\tag{4.7}
$$

Theorem 4.2. *Let M be given as above and suppose $T(\epsilon)$ and $X_{T(\epsilon)}$ are independent for ϵ small enough so that $B_m(\epsilon)$ is a normal ball. Then, for each α, $1 \le \alpha \le K$, $\kappa_\alpha = 0$ or $n_\alpha = 3$.*

Proof. Since $T(\epsilon)$ and $X_{T(\epsilon)}$ are independent, the left-hand side of (4.4) is 0. Replace f in formula (4.4) with Q_{pq} and consider the ϵ^6. By Lemma 2.6,

$$B_{pq} - \frac{B_{ij}\delta_{ij}}{n}\delta_{pq} = 0. \tag{4.8}$$

I.e., the matrix with components B_{pq} is a multiple of the identity. It follows then from (4.5) that

$$\mu \doteq \kappa_1^2(n_1-1)(n_1-3) = \kappa_2^2(n_2-1)(n_2-3) = \cdots = \kappa_K^2(n_K-1)(n_K-3). \tag{4.9}$$

If this common value $\mu = 0$, then the conclusion follows since $n_\alpha = 1$ implies $\kappa_\alpha = 0$.

If $\mu \neq 0$, replace f in formula (4.4) with Q_{pq} and consider the ϵ^8 term. By Lemma 2.6,

$$C_{pq} - \frac{C_{ij}\delta_{ij}}{n}\delta_{pq} = 0. \tag{4.10}$$

As before, the matrix C_{pq} is a multiple of the identity and substituting μ in (4.6) it follows that

$$\begin{aligned} &\mu\left(6n(n+4)(n_1+2)\kappa_1 + 7(3n+8)\sum_{\beta=1}^{K}\kappa_\beta(n_\beta-1)n_\beta\right) \\ &= \mu\left(6n(n+4)(n_2+2)\kappa_2 + 7(3n+8)\sum_{\beta=1}^{K}\kappa_\beta(n_\beta-1)n_\beta\right) \\ &= \cdots = \mu\left(6n(n+4)(n_K+2)\kappa_K + 7(3n+8)\sum_{\beta=1}^{K}\kappa_\beta(n_\beta-1)n_\beta\right). \end{aligned} \tag{4.11}$$

Assuming $\mu \neq 0$, this reduces to

$$(n_1+2)\kappa_1 = (n_2+2)\kappa_2 = \cdots = (n_K+2)\kappa_K. \tag{4.12}$$

Taking this together with (4.9) we get

$$\frac{(n_1-1)(n_1-3)}{(n_1+2)^2} = \frac{(n_2-1)(n_2-3)}{(n_2+2)^2} = \cdots = \frac{(n_K-1)(n_K-3)}{(n_K+2)^2}. \tag{4.13}$$

Now $f(x) = (x-1)(x-3)/(x+2)^2$ is strictly increasing for $x \geq 7/4$. Thus when $\mu \neq 0$, (4.13) implies that $n_1 = n_2 = \cdots = n_K$ and hence, from (4.12), $\kappa_1 = \kappa_2 = \cdots = \kappa_K$.

Now replace f in formula (4.4) with Q_{pqrs} and consider the ϵ^8 term. By Lemma 2.6,

$$D_{ijkl}Q^{pqrs}_{ijkl} = 0. \tag{4.14}$$

When $n_1 = n_2 = \cdots = n_K$ and $\kappa_1 = \kappa_2 = \cdots = \kappa_K$, (4.7) reduces to

(4.15)

$$D_{ijkl} = -\frac{\kappa_1^3(n_1-1)(n_1-3)\sum_{\alpha=1}^{K}({}_\alpha\delta_{ij})({}_\alpha\delta_{kl})}{1890(n+8)^2(n+10)} + \frac{\kappa_1^3(n_1-1)^2(n_1-3)(\delta_{ij})(\delta_{kl})}{2160(n+4)^2(n+6)}.$$

Since $(\delta_{ij})(\delta_{kl})Q^{pqrs}_{ijkl} = 0$ and $\sum_{\alpha=1}^{K}({}_\alpha\delta_{ij})({}_\alpha\delta_{kl})Q^{pqrs}_{ijkl} \neq 0$, it follows in this case also that $\kappa_1 = 0$ or $n_1 = 3$. □

5. Remarks

These results completely determine which products of constant curvature spaces have independent exit time and place for small balls. The harmonic spaces are the only other Riemannian manifolds known to satisfy this independence property. One might expect to classify more completely manifolds with this independence property by considering curvature conditions that follow from the same level of asymptotics for a general symmetric space. The computations required would be even more complicated than the special case we consider here.

We also note that due to the large number of calculations involved in the reduction process, we have used the symbolic algebra capabilities of *Mathematica*(TM) to carry out some of the calculations. *Mathematica*(TM), is a mathematical software system developed and distributed by Wolfram Research, Inc.

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