THE Baire CATEGORY THEOREM
AND THE EVASION NUMBER

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Abstract. In this paper we prove that $e \leq \text{cov}(\mathcal{M})$ where $e$ is the evasion number defined by Blass. This answers negatively a question asked by Brendle and Shelah.

1. Introduction

Let $\text{cov}(\mathcal{M})$ denote the smallest size of a family of meager sets whose union covers the real line. The combinatorial characterization for $\text{cov}(\mathcal{M})$ has been studied by Miller and Bartoszyński, and the following result is established. For $g \in \omega^\omega$, let $S^g = \prod_{n<\omega}[\omega]^{\leq g(n)}$. Each element of $S^g$ is called a slalom.

Theorem 1.1 ([1, Lemma 2.4.2]). The following cardinalities are the same:

1. $\text{cov}(\mathcal{M})$.
2. the smallest size of $F \subseteq \omega^\omega$ such that for every $h \in \omega^\omega$ there exists $f \in F$ with $f(n) \neq h(n)$ for all but finitely many $n < \omega$.
3. the smallest cardinality $\kappa$ satisfying the following: for every $g \in \omega^\omega$ there exists $F \subseteq \omega^\omega$ of size $\kappa$ such that, for all $\varphi \in S^g$, there exists $f \in F$ with $f(n) \notin \varphi(n)$ for all but finitely many $n < \omega$.

Blass [2] introduced a combinatorial concept called ‘predicting and evading’, and using this he defined the following cardinal invariant. Let $\mathcal{P}$ be the collection of functions $\pi$ from $\omega^{<\omega}$ to $\omega$. Here we call each such $\pi$ a predictor.

Definition 1.2 ([2]). The evasion number $e$ is the smallest size of $F \subseteq \omega^\omega$ such that, for every $\pi \in \mathcal{P}$ and $X \in [\omega]^\omega$, there exists $f \in F$ with $f(n) \neq \pi(f | n)$ for infinitely many $n \in X$.

Brendle and Shelah [3], [4] studied the relations between $e$ and other cardinal invariants, and asked whether $e > \text{cov}(\mathcal{M})$ is consistent [4, Subsection 3.4]. Here we give a negative answer to this question.

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2. The main result

We introduce a different form of evasion number by modifying the original definition due to Blass.

**Definition 2.1.** \(\varepsilon^*\) is the smallest size of \(F \subseteq \omega^\omega\) such that, for every \(\pi \in \mathcal{P}\) there exists \(f \in F\) with \(f(n) \neq \pi(f \upharpoonright n)\) for all but finitely many \(n < \omega\).

Clearly \(\varepsilon \leq \varepsilon^*\) holds, and it is easily seen from Theorem 1.1 that \(\text{cov}(\mathcal{M}) \leq \varepsilon^*\).

We show that \(\varepsilon^*\) gives another combinatorial characterization for \(\text{cov}(\mathcal{M})\). We prove the following theorem by modifying the proof for \(\varepsilon \leq \delta\), which is due to Blass [2, Theorem 13].

**Theorem 2.2.** \(\varepsilon^* = \text{cov}(\mathcal{M})\).

*Proof.* For a function \(h \in \omega^{\omega \times \omega}\), define \(x_h \in \omega^\omega\) recursively so that \(x_h(n) = h(n, 1 + \max\{x_h(i) : i < n\})\). Next, for a predictor \(\pi \in \mathcal{P}\), define a function \(\varphi_\pi\) from \(\omega \times \omega\) to \([\omega]^\omega\) by letting \(\varphi_\pi(n, k) = \{\pi(t) : t \in k^n\}\). By identifying \(\omega \times \omega\) with \(\omega\), we can regard \(\varphi_\pi\) as a slalom in \(S^\pi\) for a suitable \(g \in \omega^\omega\) which does not depend on \(\pi\).

Now we prove the following: for \(n < \omega\), if \(h(n, k) \notin \varphi_\pi(n, k)\) for all \(k\), then \(x_h(n) \neq \pi(x_h \upharpoonright n)\). Suppose that \(h(n, k) \notin \varphi_\pi(n, k)\) for all \(k\). Let \(k = 1 + \max\{x_h(i) : i < n\}\). Then \(x_h \upharpoonright n \in k^n\) and hence \(\pi(x_h \upharpoonright n) \in \varphi_\pi(n, k)\). On the other hand, \(x_h(n) = h(n, k) \notin \varphi_\pi(n, k)\). Thus, \(x_h(n) \neq \pi(x_h \upharpoonright n)\).

By Theorem 1.1, we can choose \(F \subseteq \omega^{\omega \times \omega}\) of size \(\text{cov}(\mathcal{M})\) so that, for each predictor \(\pi \in \mathcal{P}\), there is \(f \in F\) with \(f(n, k) \notin \varphi_\pi(n, k)\) for all but finitely many \((n, k) \in \omega \times \omega\). Then the set \(\{x_f : f \in F\}\) witnesses \(\varepsilon^* \leq \text{cov}(\mathcal{M})\), and hence \(\varepsilon^* = \text{cov}(\mathcal{M})\). \(\square\)

**Corollary 2.3.** \(\varepsilon \leq \text{cov}(\mathcal{M})\). \(\square\)

Let \(\text{non}(\mathcal{M})\) denote the smallest size of a nonmeager set of reals. We can characterize \(\text{non}(\mathcal{M})\) in a dual fashion, using [1, Lemma 2.4.8] instead of Theorem 1.1.

**Theorem 2.4.** \(\text{non}(\mathcal{M})\) is the smallest size of \(\Pi \subseteq \mathcal{P}\) satisfying the following: for every \(f \in \omega^\omega\) there exists \(\pi \in \Pi\) such that \(f(n) = \pi(f \upharpoonright n)\) for infinitely many \(n < \omega\). \(\square\)

3. A remark on the evasion ideal

Brendle [3, Subsection 3.5] introduced the notion of the evasion ideal, that is, the \(\sigma\)-ideal generated by the sets of the form

\[\{f \in \omega^\omega : f(n) = \pi(f \upharpoonright n)\text{ for all but finitely many }n \in X\}\]

for \(\pi \in \mathcal{P}\) and \(X \subseteq [\omega]^\omega\). He considered the smallest size of a subset of \(\omega^\omega\) which does not belong to this ideal.

**Definition 3.1** ([4, Definition 3.1]). \(\varepsilon(\omega)\), the uniformity of the evasion ideal, is the smallest size of \(F \subseteq \omega^\omega\) satisfying the following: for any countable family of pairs \(\{(\pi_i, X_i) : i < \omega\} \subseteq \mathcal{P} \times [\omega]^\omega\) there is \(f \in F\) such that for each \(i < \omega\) we have \(f(n) \neq \pi_i(f \upharpoonright n)\) for infinitely many \(n \in X_i\).

Clearly \(\varepsilon \leq \varepsilon(\omega)\) holds. Brendle and Shelah asked whether \(\varepsilon = \varepsilon(\omega)\) can be proved in ZFC, and they presented the following partial answer.
Theorem 3.2 ([4, Theorem 3.3]). \( \epsilon \geq \min\{\epsilon(\omega), \text{cov}(\mathcal{M})\} \). Thus either \( \epsilon < \text{cov}(\mathcal{M}) \) or \( \epsilon(\omega) \leq \text{cov}(\mathcal{M}) \) implies \( \epsilon = \epsilon(\omega) \). \( \blacksquare \)

We show that the latter assumption of the above theorem holds in ZFC.

Theorem 3.3. \( \epsilon(\omega) \leq \epsilon^* \).

Proof. Fix \( F \subseteq \omega^\omega \) of size less than \( \epsilon(\omega) \) arbitrarily. Then we can choose a countable set of pairs \( \{ (\pi_i, X_i) : i < \omega \} \subseteq \mathcal{P} \times [\omega]^\omega \) so that, for every \( f \in F \), there is \( i < \omega \) such that \( f(n) = \pi_i(f \restriction n) \) for all but finitely many \( n \in X_i \). By shrinking \( X_i \)'s if necessary, we can assume that \( X_i \)'s are pairwise disjoint. Now define \( \pi \in \mathcal{P} \) as follows: for \( t \in \omega^{<\omega} \) if \( |t| \in X_i \) for some \( i < \omega \) then \( \pi(t) = \pi_i(t) \); otherwise \( \pi(t) \) is arbitrary. Then for all \( f \in F \) we have \( f(n) = \pi(f \restriction n) \) for infinitely many \( n < \omega \).

Corollary 3.4. \( \epsilon = \epsilon(\omega) \).

Proof. By Theorems 2.2, 3.2 and 3.3. \( \blacksquare \)

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References


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