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BOURGAIN'S ANALYTIC PROJECTION REVISITED

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ABSTRACT. For a positive function w on the unit circle with $\log w \in L^1$, the following two statements are equivalent: (a) $\log w \in BMO$; (b) there is an operator Q projecting $L^p(w)$ onto $H^p(w)$ for all $1 < p < \infty$ at once and having weak type (1,1) with respect to w.

Let $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$ be the unit circle; a *weight* is a positive function w on $\mathbb{T}$ such that $\log w$ is integrable. If w is a weight, we define $H^p(w) = W^{-1/p}H^p$ $(0 < p < \infty)$, where H^p is the classical Hardy space on $\mathbb{T}$ and W is an outer function satisfying $|W| = w$ (specifically, $W = \exp(\log w + i\mathcal{H}(\log w))$, $\mathcal{H}$ being the harmonic conjugation operator).

A weight w is said to *admit an analytic projection* if there is an operator Q that projects $L^p(w)$ onto $H^p(w)$ for all $1 < p < \infty$ at once and, moreover, is of weak type (1,1) with respect to w:

$$w\{|Qf| > \lambda\} \leq \frac{C}{\lambda} \int |f|w, \qquad f \in L^1(w).$$

Furthermore, if the adjoint Q^* (relative to the duality $\langle f, g\rangle = \int f\bar{g}w$) is also of weak type (1,1) with respect to w, we say that Q is a *nice analytic projection.*

If $w \equiv 1$ (more generally, w satisfies the Muckenhoupt condition A_1), then the standard Riesz projection $\mathbb{P}\colon L^2 \to H^2, \mathbb{P}f = \sum_{n\geq 0} \hat{f}(n)z^n$, is a nice analytic projection for w. At first glance, it is difficult to imagine how to advance here beyond A_1; however, this is possible, as the following remarkable result of Bourgain [B] shows.

Theorem 1. *For every integrable weight u there exists a weight w admitting a nice analytic projection and satisfying $w \geq u$, $\int w \leq c \int u$ (c is a universal constant).*

See also the expositions of this result in [Ki1] and [W]. Of course, now the projection in question is no longer $\mathbb{P}$. Also, it should be noted that the weak type (1,1) inequality for the conjugate projection ("niceness") was mentioned in [W] for the first time, but informally the result is Bourgain's as stated.

However strong Theorem 1 is, this is an existence result leaving aside the question of a complete description of the weights w that admit a (nice) analytic projection. Here we prove that precisely the weights satisfying $\log w \in BMO$ are such. In order to give a more complete statement, we need a technical notion. A system $\{\varphi_j\}_{j\in\mathbb{Z}}$,

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$\varphi_j \in H^\infty$, is called an *analytic decomposition of unity subordinate to a weight* w if there is a constant C such that

$$|\varphi_j|^{1/8} w \le C 2^j, \;\; j \in \mathbb{Z}; \tag{1}$$

$$\sum\nolimits_{j\in\mathbb{Z}} |\varphi_j|^{1/8} 2^j \le C w; \tag{2}$$

$$\sum\nolimits_{j\in\mathbb{Z}} |\varphi_j|^{1/8} \le C; \tag{3}$$

$$\sum\nolimits_{j\in\mathbb{Z}} \varphi_j = 1. \tag{4}$$

(Later we shall see that the exponent $1/8$ is not of crucial importance.)

Theorem 2. *For a weight w, the following statements are equivalent*:

1) $\log w \in BMO$;
2) *there exists an analytic decomposition of unity subordinate to* w;
3) w *admits a nice analytic projection*;
4) w *admits an analytic projection*;
5) *there is an operator Q projecting $L^p(w)$ onto $H^p(w)$ for two different values of* p.

Moreover, if 1) *is true, all constants implicitly involved in statements* 2)–5) *can be controlled in terms of* $\|\log w\|_{BMO}$ *(and p, when applicable).*

(We remind the reader that any constant function has zero BMO-norm.)

We also include a version with two weights, which may be of interest in connection with interpolation of linear operators. As will be seen from the proof, generalization to the case of any finite collection of weights is possible.

Theorem 3. *Let u, v be two weights such that* $\log u$, $\log v \in BMO$. *Then there is an operator Q that is a nice analytic projection for u and v simultaneously.*

Before passing to the proofs, we observe that Theorem 1 is an easy consequence of Theorem 2. Indeed, if u is an integrable weight, then the function $w = M(u^{1/2})^2$ (M is the Hardy-Littlewood maximal operator) satisfies $\|w\|_{L^1} \le C\|u\|_{L^1}$ and $\|\log w\|_{BMO} \le C$ (see, e.g., [GR] for the latter).

Proof of Theorem 2. The implications 3) $\Rightarrow$ 4) $\Rightarrow$ 5) are trivial, and 5) $\Rightarrow$ 1) is [KiX, Corollary 2.2]. Next, the implication 2) $\Rightarrow$ 3) was in fact proved in [B] (see also [W] and [Ki1]); however, the arguments will be reproduced in a more general setting in the proof of Theorem 3 (see below).

Thus, the only really new statement is 1) $\Rightarrow$ 2). To prove this, we need some preparations. Let (X_0, X_1) be a compatible couple of Banach spaces (i.e., both X_0 and X_1 are continuously included in some linear topological space), and let Y_0, Y_1 be subspaces of X_0 and X_1, respectively. The couple (Y_0, Y_1) is said to be K-*closed* in (X_0, X_1) if whenever $y \in Y_0 + Y_1$ is decomposed as $y = x_0 + x_1$, $x_i \in X_i$, there is another decomposition $y = y_0 + y_1$, where $y_i \in Y_i$ and $\|y_i\|_{Y_i} \le C\|x_i\|_{X_i}$ ($i = 0, 1$; C does not depend on the vectors involved).

For a weight w, we define $L^\infty(w) = w \cdot L^\infty$ ($\|\varphi\|_{L^\infty(w)} = \|\varphi/w\|_{L^\infty}$) and $H^\infty(w) = WH^\infty$, where W is an outer function with $w = |W|$. Though this definition does not extend the scales $L^p(w)$ and $H^p(w)$ to $p = \infty$ "by continuity", it is convenient because of the relation $L^1(w)^* = L^\infty(w)$, under the duality determined by the nonweighted bilinear form $\int fg$.

Lemma 1. *If $\log w_0$, $\log w_1 \in BMO$, then the couple $(H^\infty(w_0), H^\infty(w_1))$ is K-closed in $(L^\infty(w_0), L^\infty(w_1))$, with the corresponding constant controlled in terms of $\|\log w_0\|_{BMO}$, $\|\log w_1\|_{BMO}$ only.*

For the proof, we invoke duality (as in [P]), which allows us to pass to the annihilators of the spaces constituting the smaller couple. Thus, our problem becomes reduced to the problem of K-closedness of the couple $(H^1(w_0), H^1(w_1))$ in $(L^1(w_0), L^1(w_1))$. The latter has a positive solution; see Theorem 3.1 in [KiX]. However, we also refer the reader to [Ki2] for a self-contained exposition free of some unnecessary complications of [KiX].

Now, let $\log w \in BMO$. Fixing $\varepsilon > 0$, we construct functions $\varphi_j \in H^\infty$ satisfying (1)–(4) with the exponent $1/8$ replaced by ε, and with $C = C(\varepsilon, \|\log w\|_{BMO})$. For $\lambda > 0$, we introduce two weights: $w_0 = \min\{(\lambda^{-1}w)^{2/\varepsilon}, 1\}, w_1 = \min\{(\lambda w^{-1})^{1/\varepsilon}, 1\}$. It is easily seen that

$$\|\log w_0\|_{BMO},\ \|\log w_1\|_{BMO} \le C(\varepsilon,\ \|\log w\|_{BMO})$$

(indeed, first we observe that multiplication of a weight by a constant does not change the BMO-norm of the logarithm; next, we recall that the Lipschitz functions operate on BMO). We are going to apply Lemma 1 and obtain two functions $g \in H^\infty(w_0),\ h \in H^\infty(w_1)$ such that

$$1 = g + h, \qquad |g| \le Cw_0,\ |h| \le Cw_1, \tag{5}$$

referring to the fact that $w_0 + w_1 \ge 1$, so that a measurable decomposition of 1 with properties (5) does exist. However, we should be cautious, because the inclusion $1 \in H^\infty(w_0) + H^\infty(w_1)$ does not seem to be clear *a priori*. To circumvent this difficulty, we let $E_\delta = \{|w_0| \le \delta\}$, and introduce the function u_δ equal to 1 on $\mathbb{T}\backslash E_\delta$ and to w_0 on E_δ. Next, we consider the outer function $U_\delta = \exp \Phi_\delta$, where

$$\Phi_\delta = \log u_\delta + i\mathcal{H}(\log u_\delta).$$

Then $U_\delta \in H^\infty(w_0) \subset H^\infty(w_0) + H^\infty(w_1)$ and $|U_\delta| \le w_0 + w_1$, whence, by Lemma 1, $U_\delta = g_\delta + h_\delta$ with $g_\delta \in H^\infty(w_0)$, $h_\delta \in H^\infty(w_1)$, and $|g_\delta| \le Cw_0, |h_\delta| \le Cw_1$.

Now, since $\log w_0 \in L^1$, it readily follows that $\Phi_\delta(z) \to 0$ as $\delta \to 0$ for every z with $|z| < 1$ (as usual, we denote the natural analytic extension of a boundary function to the unit disk by the same symbol), whence $U_\delta(z) \to 1$ as $\delta \to 0$ ($|z| < 1$). Let g and h be w^*-limit points ($\delta \to 0$) of the g_δ and the h_δ, respectively, in the spaces $H^\infty(w_0)$ and $H^\infty(w_1)$. It is easily seen that $1 = g(z) + h(z)$ ($|z| < 1$), and (5) follows.

We vary the parameter λ in the above construction, putting $\lambda = 2^n$, $n \in \mathbb{Z}$. For each n, we obtain two functions g_n and h_n, as in (5). Clearly, the functions $\varphi_n = g_n - g_{n+1} = h_{n+1} - h_n$ are (the boundary values of) analytic functions in the Smirnov class and

$$|\varphi_n| \le c \min\{(2^{-n}w)^{2/\varepsilon},\ (2^n w^{-1})^{1/\varepsilon}\}. \tag{6}$$

We claim that $\{\varphi_n\}$ is the desired decomposition of unity. Indeed, first

$$\sum_{k \le j \le l} \varphi_j = 1 - h_k - g_{l+1} \to 1 \quad \text{a.e.}$$

as $k \to -\infty$ and $l \to +\infty$, because $|g_{l+1}| \le C(2^{-l-1}w)^{2/\varepsilon}$ and $|h_k| \le C(2^k w^{-1})^{1/\varepsilon}$. This proves (4). Now, by (6), $|\varphi_n|^\varepsilon w \le c2^n$, which is (1). We check (2) and (3), again using (6). Let $e_k = \{2^k \le w < 2^{k+1}\}$, $k \in \mathbb{Z}$. Then

$$\begin{aligned}\sum_{n\in\mathbb{Z}} 2^n|\varphi_n|^\varepsilon &\le C\sum_{n\in\mathbb{Z}} 2^n \left(\sum_{k\le n} 2^{-2n}2^{2k}\chi_{e_k} + \sum_{k>n} 2^n 2^{-k}\chi_{e_k}\right)\\ &= C\left[\sum_{k\in\mathbb{Z}}\left(\sum_{n\ge k} 2^{-n}\right)2^{2k}\chi_{e_k} + \sum_{k\in\mathbb{Z}}\left(\sum_{n<k} 2^{2n}\right)2^{-k}\chi_{e_k}\right]\\ &\le C'\sum_{k\in\mathbb{Z}} 2^k\chi_{e_k} \le C''w,\end{aligned}$$

and

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\varphi_n|^\varepsilon &\le C\sum_{n\in\mathbb{Z}} \left(\sum_{k\le n} 2^{-2n}2^{2k}\chi_{e_n} + \sum_{k>n} 2^n 2^{-k}\chi_{e_k}\right)\\ &= C\left[\sum_{k\in\mathbb{Z}}\left(\sum_{n\ge k} 2^{-2n}\right)2^{2k}\chi_{e_k} + \sum_{k\in\mathbb{Z}}\left(\sum_{n<k} 2^{n}\right)2^{-k}\chi_{e_k}\right]\\ &\le C'\sum_{k\in\mathbb{Z}} \chi_{e_k} = C'. \quad \square\end{aligned}$$

Corollary 1. *Let G be a Smirnov domain with boundary Γ, and let φ be a conformal mapping of the unit disk onto G. Then the condition $\log|\varphi'| \in BMO$ is implied by the existence of an operator Q that projects $L^p(\Gamma)$ onto $H^p(\Gamma)$ (the boundary class for $H^p(G)$) for two different values of p at once, and implies the existence of an operator doing so for all $1 < p < \infty$ at once and, moreover, having weak type (1,1).*

Proof of Theorem 3. For the weight $w = vu^{-1}$, the implication 1) ⇒ 2) in Theorem 2 yields a collection $\{\varphi_j\} \subset H^\infty$ satisfying (3), (4) and

$$|\varphi_j|^{1/8}v \le C2^j u, \qquad j \in \mathbb{Z}; \tag{1*}$$

$$\sum_{j\in\mathbb{Z}} |\varphi_j|^{1/8}2^j u \le Cv. \tag{2*}$$

We factorize $\varphi_j = \theta_j\psi_j^8$ with θ_j inner and ψ_j outer. $\square$

Lemma 2. *Assume that there exists a nice analytic projection Q for u, and put*

$$Q_1 f = \sum_{j\in\mathbb{Z}} \theta_j\psi_j^4 Q(f\psi_j^4).$$

Then Q_1 is a nice analytic projection for u and v simultaneously.

If this is proved, we first take $u = 1$, $Q = \mathbb{P}$ (the Riesz projection) to deduce that 2) ⇒ 3) in Theorem 2. This shows that, under the assumptions of Theorem 3, a nice analytic projection does exist for u. Now another application of Lemma 2 proves Theorem 3.

In order to verify Lemma 2, we invoke the following characterization of the weak type (1,1) property (see [B] and also [Ki1], [W]).

Lemma 3. *A linear operator T acting from a subset of $L^1(\mu)$ to μ-measurable functions is of weak type* (1,1) *if and only if*

$$\int |Tf|^{1/2}|g|\,d\mu \le C\|f\|_{L^1(\mu)}^{1/2}\|g\|_{L^1(\mu)}^{1/2}\|g\|_{L^\infty(\mu)}^{1/2}$$

for every $f \in \operatorname{Dom} T$, $g \in L^1(\mu) \cap L^\infty(\mu)$.

Proof. The "only if" part is a well-known (and not difficult) Hölder type inequality. For the "if" part, take $g = \chi_e$, where e is an arbitrary subset of finite measure in $\{|Tf| < \lambda\}$. □

Proof of Lemma 2. That Q_1 fixes analytic functions is a consequence of (4).

(a) Q_1 is *$L^p(v)$-bounded*, $1 < p < \infty$. Indeed, by (3), the convexity of the function $t \mapsto t^p$, and (1*) we have

$$\begin{aligned}\int |Q_1 f|^p v &\le C\sum_j \int |Q(f\psi_j^4)|^p|\psi_j|v\\ &\le C\sum_j 2^j\int |Q(f\psi_j^4)|^p u \le C\int |f|^p\left(\sum_j 2^j|\psi_j|u\right)\\ &\le C\int |f|^p v,\end{aligned}$$

in accordance with (2*).

(b) Q_1 *is of weak type* (1,1) *with respect to* v. We take $g \in L^1(v)\cap L^\infty$ and write, applying Lemma 3 to Q:

$$\begin{aligned}\int |Q_1 f|^{1/2}|g|v &\le \sum_j \int |\psi_j|^2|Q(f\psi_j^4)|^{1/2}|g|v\\ &\le C\sum_j 2^j\int |Q(f\psi_j^4)|^{1/2}|\psi_j g|u \le C\|g\|_\infty^{1/2}\sum_j 2^j\|f\psi_j^4\|_{L^1(u)}^{1/2}\|\psi_j g\|_{L^1(u)}^{1/2}\\ &\le C\|g\|_\infty^{1/2}\left(\sum_j 2^j\|f\psi_j\|_{L^1(u)}\right)^{1/2}\left(\sum_j 2^j\|g\psi_j\|_{L^1(u)}\right)^{1/2}\\ &\le C\|g\|_\infty^{1/2}\|f\|_{L^1(v)}^{1/2}\|g\|_{L^1(v)}^{1/2},\end{aligned}$$

which implies the required statement, again by Lemma 3.

(c) We denote by "$*$" conjugation with respect to the sesquilinear form $(f,g) \mapsto \int f\bar{g}v$ and by "*ad*" conjugation with respect to the form $(f,g) \mapsto \int f\bar{g}u$. Then, as is easily seen,

$$Q_1^* h = \sum_j \bar{\psi}_j^4 Q^{ad}\left(h\bar{\theta}_j\bar{\psi}_j^4\frac{v}{u}\right)\frac{u}{v}.$$

We verify that Q_1^* *is of weak type* (1,1) *with respect to* v:

$$\begin{aligned}\int |Q_1^* h|^{1/2}|g|v &\le \sum_j \int |\psi_j|^2 v^{1/2}u^{1/2}|g| \left|Q^{ad}\left(h\bar\theta_j\bar\psi_j^4\frac{v}{u}\right)\right| \\ &\le C\sum_j \int 2^{j/2}\left|Q^{ad}\left(h\bar\theta_j\bar\psi_j^4\frac{v}{u}\right)\right|^{1/2}|g|\,|\psi_j|^{3/2}u \\ &\le C\|g\|_\infty^{1/2}\sum_j 2^{j/2}\|h\psi_j^4\frac{v}{u}\|_{L^1(u)}^{1/2}\|g|\psi_j|^{3/2}\|_{L^1(u)}^{1/2} \\ &\le C\|g\|_\infty^{1/2}\left(\sum_j \|h\psi_j^4\|_{L^1(v)}\right)^{1/2}\left(\sum_j 2^j\|g\psi_j\|_{L^1(u)}\right)^{1/2} \\ &\le C\|g\|_\infty^{1/2}\|h\|_{L^1(v)}^{1/2}\|g\|_{L^1(v)}^{1/2},\end{aligned}$$

by (3) and (2^*).

(d) That Q_1 *is a nice analytic projection for* u is proved by similar but simpler calculations, which are left to the reader. It should be kept in mind that, in order to prove Lemma 2 as stated, one must check the weak type (1,1) inequality with respect to u for Q_1 and Q_1^{ad} (*not* Q_1^*); for Q_1^{ad} we have the formula

$$Q_1^{ad}f = \sum_j \bar\psi_j^4 Q^{ad}(h\bar\theta_j\bar\psi_j^4). \quad \square$$

Remark. However, Q_1^* is also of weak type (1,1) with respect to u. Here is the calculation: by (2^*), $v^{-1/2}|\psi_j|^{1/2} \le 2^{-j/2}u^{-1/2}$, whence

$$\begin{aligned}\int |Q_1^* h|^{1/2}|g|u &\le \sum_j |\psi_j|^2 \left|Q^{ad}\left(h\bar\theta_j\overline{\psi}_j^4\frac{v}{u}\right)\right|^{1/2}|g|u^{3/2}v^{-1/2} \\ &\le C\sum_j 2^{-j/2}\int \left|Q^{ad}\left(h\bar\theta_j\bar\psi_j^4\frac{v}{u}\right)\right|^{1/2}|g|\,|\psi_j|^{3/2}u \\ &\le C\|g\|_\infty^{1/2}\sum_j 2^{-j/2}\|h\psi_j^4\frac{v}{u}\|_{L^1(u)}^{1/2}\|g|\psi_j|^{3/2}\|_{L^1(u)}^{1/2} \\ &\le C\|g\|_\infty^{1/2}\left(\sum_j 2^{-j}\|h\psi_j^4\|_{L^1(v)}\right)^{1/2}\left(\sum_j \|g|\psi_j|\,\|_{L^1(u)}\right)^{1/2} \\ &\le C\|g\|_\infty^{1/2}\|h\|_{L^1(u)}^{1/2}\|g\|_{L^1(u)}^{1/2},\end{aligned}$$

because $|\psi_j|v \le C2^j u$ and $\sum|\psi_j| \le C$. $\square$

We finish the paper with a few remarks concerning statement 2) of Theorem 2. In a quite indirect way, we have proved that if conditions (1)–(4) are satisfied for some weight w, then similar conditions can be ensured for w with an arbitrarily small exponent $\varepsilon > 0$ in place of 1/8. (For convenience, we denote these modified conditions by (1_ε), (2_ε), (3_ε); no new notation is needed for (4).)

Of course, this can be seen directly and in a more general form. We restrict ourselves to $\varepsilon \le 1$ and start with the observation that our conditions become stronger as ε decreases. In particular, (3_ε) implies that

$$\sum_{j\in\mathbb{Z}} |\varphi_j| \le C. \tag{7}$$

Now, we show that if for some $0 < \varepsilon \le 1$ we have (1_ε), (2_ε), (4) and (7), then we can construct some other functions ψ_j satisfying $(1_{\varepsilon/2})$, $(2_{\varepsilon/2})$, (4), and (7).

Indeed, we raise (4) to the power 3 to obtain

$$1 = \sum_{\substack{1\le j\le k\\ i,j,k\in\mathbb{Z}}} c_{ijk}\,\varphi_i\,\varphi_j\,\varphi_k,$$

where $|c_{ijk}| \le 6$ for all i, j, k. Fixing j, we put

$$\psi_j = \varphi_j \left(\sum_{i\le j} \sum_{k\ge j} c_{ijk}\,\varphi_i \varphi_k \right)$$

and claim that these functions satisfy the required properties.

Indeed, (4) for $\{\psi_j\}$ is clear. Next, by (7),

$$|\psi_j| \le 6|\varphi_j| \left(\sum_{i\le j}|\varphi_i|\right)\left(\sum_{k\ge j}|\varphi_k|\right) \le \begin{cases} C'|\varphi_j|\sum_{i\le j}|\varphi_i| \\ C'|\varphi_j|\sum_{k\ge j}|\varphi_k|. \end{cases}$$

Using the upper line in the latter inequality, we obtain

$$|\psi_j|^{\varepsilon/2} w \le C'(|\varphi_j|^\varepsilon w)^{1/2}\left(\sum_{i\le j}|\varphi_i| w^{1/\varepsilon}\right)^{\varepsilon/2} \le C'' 2^{j/2}\left(\sum_{i\le j} 2^{i/\varepsilon}\right)^{\varepsilon/2} \le C2^j,$$

whereas the lower line yields

$$\begin{aligned}\sum_j 2^j|\psi_j|^{\varepsilon/2} &\le C'\sum_j (2^j|\varphi_j|^\varepsilon)^{1/2}\left(\sum_{k\ge j} 2^{j/\varepsilon}|\varphi_k|\right)^{\varepsilon/2} \\ &\le C'' w^{1/2}\left(\sum_{k\in\mathbb{Z}} 2^k|\varphi_k|^\varepsilon\right)^{1/2} \le Cw.\end{aligned}$$

Finally, since $|\psi_j| \le c|\varphi_j|$, (7) persists for $\{\psi_j\}$.

Iterating, we can ensure (1_δ), (2_δ), (4), and (7) for an arbitrarily small δ. Next, we observe that (1_δ) and (2_δ) together imply $(3_{2\delta})$:

$$\sum |\psi_j|^{2\delta} w \le c \sum |\psi_j|^\delta 2^j \le c' w.$$

Now, we introduce two definitions. Let X be a Banach lattice of measurable functions (synonyms: a Banach ideal space, a Köthe function space) on $\mathbb{T}$.

Definition 1 (cf. [KiX]). X is said to *admit sufficiently many analytic decompositions of unity* if for every $a \in X$ there exists a weight $w \ge a$ such that $w \in X$, $\|w\|_X \le C\|a\|_X$, and there is an analytic decomposition of unity with $\varepsilon = 1/4$ subordinated to w (all constants involved explicitly and implicitly must not depend on a).

Definition 2 (see [Ka], [Ki2]). X is said to be *BMO-regular* if for every $a \in X$ there exists a weight $w \geq |a|$ such that $w \in X$, $\|w\|_X \leq C\|a\|_X$, and $\|\log w\|_{BMO} \leq C$ (C is independent of a).

Both notions have been used in the study of certain interpolation problems for Hardy-type subspaces in lattices of measurable functions. (Definition 1 is given here in the spirit rather than along the lines of [KiX], where only some particular cases were considered—however, sometimes in a more general setting of function spaces on $(\mathbb{T}, \Omega, m \times \mu)$, where (Ω, μ) is a measure space.) In the end, the second notion has turned out to be far more convenient technically than the first. The preceding discussion clarifies the formal relationship between them.

Corollary 2. *Definitions 1 and 2 describe one and the same class of lattices.*

Proof. We have seen that we may replace $\varepsilon = 1/4$ with $\varepsilon = 1/8$ in Definition 1. After this we apply Theorem 2. □ □

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