INTEGER SETS WITH DISTINCT SUBSET SUMS

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(Communicated by David E. Rohrlich)

Abstract. We give a simple, elementary new proof of a generalization of the following conjecture of Paul Erdős: the sum of the elements of a finite integer set with distinct subset sums is less than 2.

Let \( a_0 < a_1 < \cdots < a_n \) be positive integers with all the sums \( \sum_{i=0}^{n} \varepsilon_i a_i (\varepsilon_i = 0; 1) \) different. It was conjectured by P. Erdős and proved by C. Ryavec that then

\[
\sum_{i=0}^{n} \frac{1}{a_i} < 2 \left( = \sum_{i=0}^{\infty} \frac{1}{2^i} \right)
\]

(see [1]). F. Hanson, J. M. Steele and F. Stenger [2] proved the generalization

\[
\sum_{i=0}^{n} \frac{1}{a_i^s} < \frac{1}{1 - 2^{-s}} \left( = \sum_{i=0}^{\infty} \frac{1}{2^{is}} \right)
\]

for all real \( s > 0 \). These proofs are relatively simple but use generating functions and other methods in analysis. I have recently learned that a brilliant elementary solution to Erdős’s original problem was found by A. Bruen and D. Borwein, more than 20 years ago. See [3] or [4].

We prove by elementary methods the more general statement that (continuing to assume that all sums \( \sum_{i=0}^{n} \varepsilon_i a_i \) are different)

\[
\sum_{i=0}^{n} f(a_i) \leq \sum_{i=0}^{n} f(2^i) \tag{1}
\]

for any convex decreasing function \( f \).

The hypothesis implies for \( k = 0; 1; \ldots; n \) that

\[
\sum_{i=0}^{k} a_i \geq 2^{k+1} - 1 \tag{*}
\]

since there exist \( 2^{k+1} - 1 \) distinct positive integers (namely, \( \sum_{i=0}^{k} \varepsilon_i a_i (\varepsilon_i = 0; 1, (\varepsilon_i)_0^k \neq (0)_k^k) \)) which are all less than or equal to \( \sum_{i=0}^{k} a_i \).

Consider all \( (n+1) \)-tuples of positive integers \( a_0 < a_1 < \cdots < a_n \) having property \( (*) \) for \( k = 0; 1; \ldots; n \). It suffices to prove that among all these, the \( (n+1) \)-tuple
$a_i = 2^i \ (i = 0; 1; \ldots ; n)$ has maximal $\sum_{i=0}^{n} f(a_i)$. Consider any such $(n + 1)$-tuple. We define an index $k$ to be good if equality holds in $(\ast)$ and bad otherwise. If all indices are good then clearly $a_i = 2^i \ (i = 0; 1; \ldots ; n)$. If not, then let $p$ be the smallest bad index. If there is any good index larger than $p$, then let $q$ be the smallest such index. Since $a_i = 2^i$ for $i < p$ and $a_p > 2^p$, it follows that the number $a_p - 1$ is a positive integer and does not occur among the numbers $a_i$. If $q$ exists, then $q \neq 0$ and so

$$a_q = \sum_{i=0}^{q} a_i - \sum_{i=0}^{q-1} a_i < 2^{q+1} - 1 - (2^q - 1) = 2^q,$$

since $q-1$ is bad and $q$ is good. If $q \leq n-1$, then $a_{q+1} = \sum_{i=0}^{q} a_i - \sum_{i=0}^{q} a_i \geq 2^{q+1}$, hence $a_{q+1} > a_q$ and so the number $a_q + 1$ does not occur among the numbers $a_i$.

Therefore, we may replace $a_p$ by $a_p - 1$ and, if $q$ exists, $a_q$ by $a_q + 1$. The property $1 \leq a_0 < a_1 < \cdots < a_n$ and the property $(\ast)$ will be preserved (this follows from the definition of $p$ and $q$). Since $f$ is decreasing and convex, the sum $\sum_{i=0}^{n} f(a_i)$ will not be decreased whether $q$ exists or not.

We may repeat this procedure until we reach the $(n + 1)$-tuple $a_i = 2^i$. This will happen after a finite number of steps since the sum $\sum_{i=0}^{n} (n + 1 - i) a_i$ takes only positive integer values and is decreased by at least 1 in every step. This completes the proof.

It is easily seen that if $f$ is strictly decreasing and strictly convex (as in the case $f(x) = x^{-s} (s > 0)$), then equality in (1) holds only for $a_i = 2^i \ (i = 0; 1; \ldots ; n)$.

REFERENCES


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