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INTEGER SETS WITH DISTINCT SUBSET SUMS

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Abstract. We give a simple, elementary new proof of a generalization of the following conjecture of Paul Erdős: the sum of the elements of a finite integer set with distinct subset sums is less than 2.

Let $a_0 < a_1 < \cdots < a_n$ be positive integers with all the sums $\sum_{i=0}^{n} \varepsilon_i a_i$ ($\varepsilon_i = 0; 1$) different. It was conjectured by P. Erdős and proved by C. Ryavec that then

$$\sum_{i=0}^{n} \frac{1}{a_i} < 2 \left(= \sum_{i=0}^{\infty} \frac{1}{2^i} \right)$$

(see [1]). F. Hanson, J. M. Steele and F. Stenger [2] proved the generalization

$$\sum_{i=0}^{n} \frac{1}{a_i^s} < \frac{1}{1 - 2^{-s}} \left(= \sum_{i=0}^{\infty} \frac{1}{2^{is}} \right)$$

for all real $s > 0$. These proofs are relatively simple but use generating functions and other methods in analysis. I have recently learned that a brilliant elementary solution to Erdős's original problem was found by A. Bruen and D. Borwein, more than 20 years ago. See [3] or [4].

We prove by elementary methods the more general statement that (continuing to assume that all sums $\sum_{i=0}^{n} \varepsilon_i a_i$ are different)

$$\sum_{i=0}^{n} f(a_i) \le \sum_{i=0}^{n} f(2^i) \tag{1}$$

for any convex decreasing function f.

The hypothesis implies for $k = 0; 1; \dots ; n$ that

$$\sum_{i=0}^{k} a_i \ge 2^{k+1} - 1 \tag{$*$}$$

since there exist $2^{k+1} - 1$ distinct positive integers (namely, $\sum_{i=0}^{k} \varepsilon_i a_i$ ($\varepsilon_i = 0; 1$, $(\varepsilon_i)_0^k \neq (0)_0^k$)) which are all less than or equal to $\sum_{i=0}^{k} a_i$.

Consider all $(n+1)$-tuples of positive integers $a_0 < a_1 < \cdots < a_n$ having property $(*)$ for $k = 0; 1; \dots ; n$. It suffices to prove that among all these, the $(n + 1)$-tuple

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$a_i = 2^i$ $(i = 0; 1; \ldots ; n)$ has maximal $\sum_{i=0}^{n} f(a_i)$. Consider any such $(n+1)$-tuple. We define an index k to be good if equality holds in $(*)$ and bad otherwise. If all indices are good then clearly $a_i = 2^i$ $(i = 0; 1; \ldots ; n)$. If not, then let p be the smallest bad index. If there is any good index larger than p, then let q be the smallest such index. Since $a_i = 2^i$ for $i < p$ and $a_p > 2^p$, it follows that the number $a_p - 1$ is a positive integer and does not occur among the numbers a_i. If q exists, then $q \neq 0$ and so

$$a_q = \sum_{i=0}^{q} a_i - \sum_{i=0}^{q-1} a_i < 2^{q+1} - 1 - (2^q - 1) = 2^q,$$

since $q-1$ is bad and q is good. If $q \leq n-1$, then $a_{q+1} = \sum_{i=0}^{q+1} a_i - \sum_{i=0}^{q} a_i \geq 2^{q+1}$, hence $a_q + 1 < a_{q+1}$ and so the number $a_q + 1$ does not occur among the numbers a_i.

Therefore, we may replace a_p by $a_p - 1$ and, if q exists, a_q by $a_q + 1$. The property $1 \leq a_0 < a_1 < \cdots < a_n$ and the property $(*)$ will be preserved (this follows from the definition of p and q). Since f is decreasing and convex, the sum $\sum_{i=0}^{n} f(a_i)$ will not be decreased whether q exists or not.

We may repeat this procedure until we reach the $(n+1)$-tuple $a_i = 2^i$. This will happen after a finite number of steps since the sum $\sum_{i=0}^{n} (n+1-i)a_i$ takes only positive integer values and is decreased by at least 1 in every step. This completes the proof.

It is easily seen that if f is strictly decreasing and strictly convex (as in the case $f(x) = x^{-s} (s > 0)$), then equality in (1) holds only for $a_i = 2^i$ $(i = 0; 1; \ldots , n)$.

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