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A SHORT PROOF FOR THE STABILITY THEOREM FOR POSITIVE SEMIGROUPS ON $L_p(\mu)$

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Abstract. We give a short proof showing that the growth bound of a positive semigroup on $L_p(\mu)$ equals the spectral bound of its generator. It is based on a new boundedness theorem for positive convolution operators on $L_p(L_q)$. We also give a counterexample, showing that Gearhart's result does not extend from Hilbert spaces to $L_p(\mu)$-spaces.

1. The results

Let T_t be a c_0-semigroup on $L_p(\Omega, \mu)$, $1 \le p < \infty$, with generator A. T_t is called positive if $f \ge 0$ implies $T_t f \ge 0$ for all t. The spectral bound $s(A)$ of A is defined by

$$s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$$

and the growth bound of T_t is given by

$$\omega(T_t) = \inf\{\omega : \exists C < \infty \text{ with } \|T_t\| \le Ce^{\omega t} \text{ for all } t \ge 0\}.$$

The following theorem was proved in [9].

Theorem 1. *If T_t is a positive c_0-semigroup on $L_p(\Omega, \mu)$, $1 \le p < \infty$, then $s(A) = \omega(T_t)$.*

The case $p = 2$ is due to Gearhart and Greiner-Nagel, the case $p = 1$ is due to Derndinger (see [7], [8], or [3], Theorems 9.5 and 9.7), but the general case remained an open problem for about 10 years. The proof in [9] used a new spectral mapping theorem for the evolutionary semigroup $I \otimes T_t$ on $L_q(L_p)$ by Latushkin and Montgomery-Smith [5] and an extrapolation procedure for the Yosida approximation of T_t. In [6] S. Montgomery-Smith simplified the proof by replacing the extrapolation procedure by a direct resolvent estimate.

In this note we give a new and simpler proof of Theorem 1 that is based on a boundedness result for positive convolutions on mixed norm spaces $L_p(L_q)$, and which may be of independent interest (see Theorem 2 below). With this convolution result we can reduce Theorem 1 to a well-known characterization of the spectral bound in terms of weak integrability ([3], Theorem 7.4).

Finally, we point out that recent counterexamples concerning stability of semigroups (see e.g. [1]) can be "transplanted" onto L_p-spaces. At the end of this note

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we give an example of a semigroup T_t on $L_p(0,1),\ 1<p<\infty,\ p\neq 2$, for which $s(A)=s_\infty(A)<\omega(T_t)$. Here

$$s_\infty(A)=\inf\{\omega:\ R(\lambda,A) \text{ is uniformly bounded for } \lambda,\ \operatorname{Re}\lambda\geq\omega\}.$$

This shows that Gearhart's spectral mapping theorem on Hilbert space (see e.g. [3], Theorem 9.6) does not extend to L_p-spaces for $1<p<\infty,\ p\neq 2$, and answers negatively question (IV) on page 147 of [8].

To state our convolution result we need the following notation. For $1\leq p,q<\infty$ and $f\in L_{1,\mathrm{loc}}(\mathbb{R}_+\times\Omega)$ put

$$\|f\|_{p,q}=\left(\int_\Omega\left(\int_{\mathbb{R}}|f(t,\omega)|^q dt\right)^{p/q}d\mu(\omega)\right)^{1/p},$$
$$L_p(L_q)=\{f\in L_{1,\mathrm{loc}}(\mathbb{R}_+,\Omega):\ \|f\|_{p,q}<\infty\}.$$

Note that for $p=q$ we have by Fubini's theorem $L_p(L_p)=L_p(\mathbb{R},L_p(\Omega,\mu))$. For these mixed norm spaces we have the following convolution result:

Theorem 2. *For a fixed $1\leq p<\infty$, let $t\in\mathbb{R}\to K(t)$ be a function of positive operators on $L_p(\Omega,\mu)$ such that $t\to K(t)f$ is locally Bochner integrable for $f\in L_p(\Omega,\mu)$. Assume that for all $0\leq h\in L_p(\Omega,\mu)$ and $0\leq g\in L_{p'}(\Omega,\mu)$ we have*

$$\int_{\mathbb{R}}\langle g,K(t)h\rangle_{L_p}dt\leq C\|g\|_{L_{p'}}\cdot\|h\|_{L_p}.$$

Then the convolution integral

$$\mathcal{K}f(t)=\int_{-\infty}^{\infty}K(t-s)(f(s))ds$$

defined for stepfunctions $f:\mathbb{R}\to L_p(\Omega)$ extends to a bounded operator on $L_p(L_q)$ with $\|\mathcal{K}f\|_{p,q}\leq C\|f\|_{p,q}$ for all $1\leq q\leq\infty$.

2. The proofs

Proof of Theorem 1. Since $s(A)\leq\omega(T_t)$ is always true, we only have to show that $s(A)<0$ implies $\omega(T_t)<0$, or, by a result of Pazy ([3], Proposition 9.4), that $s(A)<0$ implies that for all $f\in L_p(\Omega,\mu)$

$$\int_0^\infty\|T_tf\|_{L_p}^p dt<\infty. \tag{1}$$

This claim can be reformulated as a convolution estimate. Indeed, for a fixed $\alpha>\omega(T_t)$

$$\int_0^t T_{t-s}(e^{-\alpha s}T_sf)ds=\frac{1}{\alpha}(1-e^{-\alpha t})T_tx.$$

Put $K(t)=T_t$ for $t\geq 0$ and $K(t)=0$ for $t<\infty$, and $f(t)=e^{-\alpha t}T_tf$ for $t\geq 0$ and $f(t)=0$ for $t<0$. Then for $t\geq 1$ there is a constant D such that

$$\|T_tx\|\leq D\left\|\int_{-\infty}^{\infty}K(t-s)(f(s))ds\right\|. \tag{2}$$

The function $t\to K(t)$ satisfies the assumption of Theorem 2 since by Theorem 7.4 of [3] we have for all $0\leq g\in L_{p'}$ and $0\leq f\in L_p$ that

$$\int_0^\infty\langle g,T_tf\rangle dt\leq\|R(0,A)\|\,\|g\|_{L_{p'}}\|f\|_{L_p}.$$

Since $f \in L_p(\mathbb{R}, L_p(\Omega))$ we obtain from Theorem 2

$$\int \left\| \int K(t-s)(f(s))ds \right\|_{L_p}^p d(t) \le C \int \|f(s)\|_{L_p}^p ds \le C_1 \|f\|^p. \tag{3}$$

Now (3) together with (2) implies (1) and the proof is complete. Alternatively, one can obtain (3) from the estimate in the proof of Theorem 1 in [6]. □

Proof of Theorem 2. First we check the claim for $q = 1$. Given a stepfunction $f : \mathbb{R} \to L_p(\Omega)$ with $f \ge 0$ and a $0 \le g \in L_{p'}(\Omega)$ we have for all $N \in \mathbb{N}$

$$\begin{aligned}\left\langle g, \int_{-N}^{N} \mathcal{K} f(t)dt \right\rangle_{L_p} &= \int_{-N}^{N} \langle g, \mathcal{K} f(t)\rangle_{L_p} dt \\ &= \int_{-N}^{N} \left\langle g, \int K(s)f(t-s)ds \right\rangle dt \\ &= \int \int_{-N}^{N} \langle g, K(s)f(t-s)\rangle dt\, ds \\ &= \int \left\langle g, K(s) \left[\int_{-N}^{N} f(t-s)dt \right] \right\rangle ds \\ &\le \int \langle g, K(s)h\rangle ds \le C\|g\|_{L_{p'}} \cdot \|h\|_{L_p}\end{aligned}$$

by assumption, where $h = \int f(t)dt$ with $\|h\|_{L_p} = \|f\|_{p,1}$. Since such stepfunctions are dense in $L_p(L_1)$ we can extend $\mathcal{K}$ to $L_p(L_1)$ with

$$\|\mathcal{K} f\|_{p,1} \le C\|f\|_{p,1}.$$

For $q = \infty$ and a stepfunction $f : \Omega \to L_\infty(\mathbb{R}),\ f(t,\omega) = \sum_k g_k(t)\chi_{A_k}(\omega)$, with $f \ge 0$ the integral

$$\int_{-N}^{N} K(s)f(t-s)ds = \sum_k \int_{-N}^{N} g_k(t-s)K(s)[\chi_{A_k}(\omega)]ds$$

exists. Put $h(\omega) = \operatorname{ess\,sup}_t f(t,\omega)$ with $\|h\|_{L_p} = \|f\|_{p,\infty}$. For all $g \in L_{p'}$ with $g \ge 0$, $\|g\|_{L_{p'}} = 1$ and $N \in \mathbb{N}$ we have

$$\begin{aligned}\left\langle g, \int_{-N}^{N} K(s)f(t-s)ds \right\rangle \le \left\langle g, \int_{-N}^{N} K(s)hds \right\rangle_{L_p} &\le \int \langle g, K(s)h\rangle ds \\ &\le C\|g\|_{L_{p'}} \cdot \|h\|_{L_p} = C\|f\|_{p,\infty}.\end{aligned}$$

Since stepfunctions with countably many values are dense in $L_p(L_\infty)$ we can extend $\mathcal{K}$ to $L_p(L_\infty)$ by Fatou's Lemma and continuity so that $\|\mathcal{K} f\|_{p,\infty} \le C\|f\|_{p,\infty}$.

Interpolating in the scale $L_p(L_q)$, $1 \le q \le \infty$, gives the general claim according to [2], Theorem 5.1.2. □

Example. Let $X = L_p(1,\infty) \cap L_2(1,\infty)$ with norm $\|f\| = \|f\|_2 + \|f\|_p$ for $2 < p < \infty$. Consider the semigroup $(S_t f)(x) = f(xe^t)$, $t \ge 0$, with generator $(Bf)(x) = x(\frac{d}{dx} f)(x)$ on a suitable domain and $(R(0,B)f)(x) = \int_x^\infty f(y)\frac{dy}{y}$. One can check that $s(B) = -\frac{1}{2} < -\frac{1}{p} = \omega(S_t)$ (cf. [1]). Since S_t is positive, we also have $s_\infty(B) = s(B)$ (see [3], Corollary 7.5).

By [4], 2.e.8(ii) and section 2.f, there is an isomorphism J of X onto $L_p[0,1]$ (essentially given by a stochastic integral with respect to the Poisson process). Then the semigroup $T_t = JS_tJ^{-1}$ on $L_p[0,1]$ with generator $A = JBJ^{-1}$ on $D(A) = J(D(B))$ still satisfies $s_\infty(A) = -\frac{1}{2} < -\frac{1}{p} = \omega(T_t)$.

If $1 < p < 2$ we take the dual of T_t on $L_{p'}$, to obtain a similar example.

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