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ON THE RECURSIVE SEQUENCE $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$

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Abstract. We show that every positive solution of the equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \qquad n = 0, 1, \dots ,$$

where $A \in (0, \infty)$, converges to a period two solution.

1. Introduction

Our aim in this paper is to establish that every positive solution of the equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \qquad n = 0, 1, \dots , \tag{1}$$

where

$$x_{-2},\ x_{-1},\ x_0,\ A \ \in \ (0, \infty), \tag{2}$$

converges to a period two solution. This confirms conjecture 2.4.2 in [1]. More precisely, we show that each of the two subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of every positive solution of Eq. (1) converges to a finite limit.

One of the main ingredients in our proof, and an interesting result in its own right, is to show that every positive solution of the equation

$$y_{n+1} = A + \frac{y_n}{y_{n-1}}, \qquad n = 0, 1, \dots , \tag{3}$$

where

$$y_{-1},\ y_0,\ A \ \in (0, \infty), \tag{4}$$

converges to its equilibrium $\bar{y} = 1 + A$. From this it follows easily that the equilibrium $1 + A$ of Eq. (3) is globally asymptotically stable.

It follows from the work in [2] that every positive solution of Eq. (1) is bounded and persists. It was also observed in [2] that for every real number q, Eq. (1) possesses the period two solution

$$q,\ \frac{A+1}{q},\ q,\ \frac{A+1}{q},\ \dots .$$

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We say that a solution $\{x_n\}$ of a difference equation is bounded and persists if there exist positive constants P and Q such that

$$P \leq x_n \leq Q \quad \text{for} \quad n = -1, 0, \ldots .$$

A positive semicycle of a solution $\{y_n\}$ of Eq. (3) consists of a "string" of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all greater than or equal to the equilibrium $\bar{y}$, with $l \geq -1$ and $m \leq \infty$, and such that

$$\text{either } l = -1, \quad \text{or } l > -1 \;\text{ and } y_{l-1} < \bar{y}$$

and

$$\text{either } m = \infty, \quad \text{or } m < \infty \text{ and } y_{m+1} < \bar{y}.$$

A negative semicycle of a solution $\{y_n\}$ of Eq. (3) consists of a "string" of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all less than the equilibrium $\bar{y}$, with $l \geq -1$ and $m \leq \infty$, and such that

$$\text{either } l = -1, \quad \text{or } l > -1 \;\text{ and } y_{l-1} \geq \bar{y}$$

and

$$\text{either } m = \infty, \quad \text{or } m < \infty \;\text{ and } y_{m+1} \geq \bar{y}.$$

The first semicycle of a solution starts with the term y_{-1}, and is positive if $y_{-1} \geq \bar{y}$ and negative if $y_{-1} < \bar{y}$.

2. Global asymptotic stability of Eq. (3)

Let $\{x_n\}$ be a positive solution of Eq. (1), and set

$$y_n = x_n x_{n-1}.$$

Then clearly, $\{y_n\}$ satisfies Eq. (3), and condition (4) holds. The main result in this section is the following.

Theorem 1. *Assume that* (4) *holds. Then the positive equilibrium* $\bar{y} = 1 + A$ *of Eq.* (3) *is globally aymptotically stable.*

Before we establish the above theorem we need the following result.

Lemma 1. *Let* $\{y_n\}$ *be a nontrivial positive solution of Eq.* (3)*. Then the following statements are true.*

(a) $\{y_n\}$ *oscillates about the equilibrium* $\bar{y} = 1 + A$ *with semicycles of length two or three.*

(b) The extreme in a semicycle occurs in the first or second term.

(c) For $n > 2$,

$$A < y_n < A + \frac{A+1}{A}.$$

Proof. We will first show that every positive semicycle, except possibly the first, has two or three terms. The case for the negative semicycles is similar and is omitted. Let $y_N \geq \bar{y}$ be the first term in a positive semicycle, other than the first positive semicycle. Then $y_{N-1} < \bar{y}$, and

$$y_{N+1} = A + \frac{y_N}{y_{N-1}} > A + \frac{y_N}{\bar{y}} = A + \frac{y_N}{1+A} \geq \bar{y}.$$

Now if $y_{N+1} > y_N$, then

$$y_{N+2} = A + \frac{y_{N+1}}{y_N} > A + 1 = \bar{y}.$$

Also

$$y_{N+2} = A + \frac{y_{N+1}}{y_N} \le A + \frac{y_{N+1}}{\bar{y}} = A + \frac{y_{N+1}}{1+A} < y_{N+1}.$$

Thus $\bar{y} < y_{N+2} < y_{N+1}$, and so

$$y_{N+3} = A + \frac{y_{N+2}}{y_{N+1}} < A + 1 = \bar{y},$$

and the positive semicycle has length three. If $y_{N+1} \le y_N$, then

$$y_{N+2} = A + \frac{y_{N+1}}{y_N} \le A + 1 = \bar{y},$$

and the positive semicycle has length at most three, with length equal to two unless $y_{N+1} = y_N$.

From this it is clear that every solution $\{y_n\}$ of Eq. (3) oscillates about $\bar{y} = 1+A$. It is also clear from the above that the extreme in a semicycle occurs in the first or second term.

From Eq. (3) one can see that $y_n > A$ for $n > 0$. Let y_N, where $N > 2$, be the first term in a positive semicycle. Then $A < y_{N-1} < \bar{y}$ and $A < y_{N-2} < \bar{y}$, giving

$$y_N = A + \frac{y_{N-1}}{y_{N-2}} < A + \frac{\bar{y}}{A} = A + \frac{1+A}{A}.$$

Now

$$y_{N+1} = A + \frac{y_N}{y_{N-1}} = A + \frac{A + \frac{y_{N-1}}{y_{N-2}}}{y_{N-1}} = A + \frac{A}{y_{N-1}} + \frac{1}{y_{N-2}} < A + \frac{1+A}{A}.$$

The proof is complete. □

Proof of Theorem 1. It is easy to see (by linearized stability analysis) that $\bar{y} = 1+A$ is locally asymptotically stable. So it remains to show that if $\{y_n\}$ is a nontrivial solution of Eq. (3), then

$$\lim_{n\to\infty} y_n = 1 + A.$$

To this end, define the sequences $\{L_n\}$ and $\{U_n\}$ as follows:

$$L_1 = A, \;\; U_1 = A + \frac{1+A}{A},$$

and for $n = 1, 2, ...$

$$L_{n+1} = A + \frac{1+A}{U_n} \;\text{ and }\; U_{n+1} = A + \frac{1+A}{L_{n+1}}.$$

Now, it can be seen that $\{U_n\}$ and $\{L_n\}$ are sequences of upper and lower bounds for the semicycles of the solutions $\{y_n\}$ of Eq. (3). Also

$$L_{n+1} = A + \frac{1+A}{A + \frac{1+A}{L_n}}$$

and

$$U_{n+1} = A + \frac{1+A}{A + \frac{1+A}{U_n}}.$$

From this it follows that

$$L_1 < L_2 < ... < L_n < L_{n+1} < ... < \bar{y} < ... < U_{n+1} < U_n < ... < U_2 < U_1.$$

Thus

$$\lim_{n\to\infty} L_n = L \le \bar{y} \;\text{ and }\; \lim_{n\to\infty} U_n = U \ge \bar{y}.$$

But since the only solution to

$$y = A + \frac{1+A}{A + \frac{1+A}{y}}$$

is $y = \bar{y}$, it must be true that $U = L = \bar{y}$. The proof is complete. $\square$

3. Periodic character of the solutions of Eq. (1)

It follows from Eq. (1) that

$$x_{2n+1} = \frac{A}{x_{2n}} + \frac{1}{x_{2n-2}}, \quad n = 0, 1, ..., \tag{5}$$

and

$$x_{2n+2} = \frac{A}{x_{2n+1}} + \frac{1}{x_{2n-1}}, \quad n = 0, 1, ..., \tag{6}$$

and so

$$x_{2n+2} = \frac{A}{\frac{A}{x_{2n}} + \frac{1}{x_{2n-2}}} + \frac{1}{\frac{A}{x_{2n-2}} + \frac{1}{x_{2n-4}}}, \quad n = 1, 2, ... \tag{7}$$

The following result summarizes some of the properties of the solutions of Eq. (1)

Lemma 2. *Let $\{x_n\}$ be a positive solution of Eq. (1). Then the following statements are true.*

(i) For $N \geq 0$, let

$$m_N = \min\{x_{2N-2}, x_{2N}, x_{2N+2}\}$$

and

$$M_N = \max\{x_{2N-2}, x_{2N}, x_{2N+2}\}.$$

Then

$$m_N \leq x_{2n} \leq M_N \;\; for \;\; n \geq N.$$

(ii) There exist positive numbers m and M such that $m \leq x_n \leq M$ for $n = 0, 1, ...$

(iii)

$$\lim_{n\to\infty} \frac{x_{2n}}{x_{2n-2}} = 1.$$

Proof. To prove (i), note that the function

$$f(x, y, z) = \frac{A}{\frac{A}{x} + \frac{1}{y}} + \frac{1}{\frac{A}{y} + \frac{1}{z}}$$

is increasing in x, y, and z. Thus

$$x_{2N+4} = \frac{A}{\frac{A}{x_{2N+2}} + \frac{1}{x_{2N}}} + \frac{1}{\frac{A}{x_{2N}} + \frac{1}{x_{2N-2}}} \leq \frac{A}{\frac{A}{M_N} + \frac{1}{M_N}} + \frac{1}{\frac{A}{M_N} + \frac{1}{M_N}} = M_N$$

and by induction, $x_{2n} \leq M_N$ for $n \geq N$. Similarly $x_{2n} \geq m_N$ for $n \geq N$. The proof of (ii) follows directly from (i), since (i) implies that the sequences of even and odd terms of Eq. (3) are bounded. To prove (iii), note that $y_{2n} = x_{2n}x_{2n-1}$ converges to $1 + A$, and so

$$\frac{y_{2n}}{y_{2n-1}} = \frac{x_{2n}x_{2n-1}}{x_{2n-1}x_{2n-2}} = \frac{x_{2n}}{x_{2n-2}}$$

converges to 1. Thus

$$\lim_{n\to\infty} \frac{x_{2n}}{x_{2n-2}} = 1.$$

□

The main result in this paper is the following:

Theorem 2. *Let* $\{x_n\}$ *be a positive solution of Eq.* (1)*. Then there exist positive constants* L_E *and* L_O *such that* $L_E L_O = 1 + A$ *and*

$$\lim_{n\to\infty} x_{2n} = L_E \text{ and } \lim_{n\to\infty} x_{2n+1} = L_O.$$

Proof. It follows from Eq. (5) and Lemma 2 that

$$\left| \frac{x_{2n}}{x_{2n-2}} - 1 \right| \geq \left| \frac{x_{2n} - x_{2n-2}}{M} \right| .$$

Hence

$$\lim_{n\to\infty} \left| x_{2n} - x_{2n-2} \right| = 0.$$

It is now clear from Lemma 2 (i) and (iii) that $\lim_{n\to\infty} x_{2n}$ exists and is a positive number L_E. From Eq. (5) it then follows that $\lim_{n\to\infty} x_{2n+1}$ also exists and is a positive number L_O. Finally, from Eq. (5) we see that

$$L_E L_O = 1 + A,$$

and the proof is complete. □

References

[1] G. Ladas, Open Problems and Conjectures, Journal of Difference Equations and Applications 2 (1996), 449-452.

[2] Ch. G. Philos, I. K. Purnaras and Y. G. Sficas, Global attractivity in a nonlinear difference equation, Applied Mathematics and Computers 62, (1994), 249 - 258. MR **95h:**39008

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