PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 11, November 1998, Pages 3133–3135
S 0002-9939(98)04675-9

A NOTE ON σ-SUMMABLE GROUPS

PAUL HILL

(Communicated by Ken Goodearl)

Abstract. We answer questions raised by P. Danchev in a recent paper in these *Proceedings*. It is shown that a σ-summable abelian p-group is not determined by its socle, that is, two such groups can have isometric socles without being isomorphic. It is also demonstrated that σ-summability plays essentially no role in regard to the question of whether or not $V(G)/G$ is totally projective, where $V(G)$ denotes the group of normalized units of the group algebra $F(G)$ with F being a perfect field of characteristic p.

The purpose of this brief note is to answer two questions raised in [D] about σ-summable abelian groups. Also, we address the question that is begged by the statement in [D] that some subgroups and direct sums of σ-summable groups are σ-summable; the word "some" was omitted in the preprint version.

The class of σ-summable groups was introduced in [LM]. A p-primary abelian group G is σ-summable if $G[p]$ is the union of an ascending sequence of subsocles S_n having the property that the heights of the nonzero elements of S_n, computed in G, are bounded by some ordinal λ_n less than the length λ of G. The following proposition states an equivalent definition in terms of subgroups instead of subsocles.

Proposition 1. *An abelian p-group G is σ-summable if and only if G is the union of an ascending sequence of subgroups H_n with the property that the heights of the nonzero elements of H_n, computed in G, are bounded by some ordinal λ_n less than the length λ of G.*

Proof. If G is the union of an ascending sequence of subgroups H_n satisfying the prescribed conditions, we only need to let $S_n = H_n[p]$ in order to demonstrate that G is, by definition, σ-summable.

Conversely, if G is σ-summable and the ascending sequence of subsocles S_n leading up to $G[p]$ is a manifestation of this, let H_0 be maximal in G with respect to $H_0[p] = S_0$. Inductively, choose H_{n+1} so that it is maximal with respect to the following conditions: (1) $H_{n+1} \supseteq H_n$, and (2) $H_{n+1}[p] = S_{n+1}$. In this connection, it is noted that $(H_n + S_{n+1})[p] = S_{n+1}$. Hence, the set to which we are applying Zorn's lemma is nonempty. Let H be the union of the ascending sequence of subgroups H_n. The maximality of H_n having the prescribed socle S_n implies that $pG \cap H_n = pH_n$. Clearly, H retains this property, $pG \cap H = pH$. Since $H[p] = G[p]$, the union of the S_n's, we conclude that $G = H$. Thus G is the union of the H_n's.

Received by the editors February 18, 1997.
1991 *Mathematics Subject Classification.* Primary 20K10, 20K07.

But if λ_n is a bound for the heights of the nonzero elements of S_n, then it must be a bound also for the heights of all the nonzero elements of H_n as well.

Note that a σ-summable group must be reduced, that is, its divisible component is zero. An immediate consequence of the preceding proposition is the following.

Corollary. *If the reduced p-group G is the union of an ascending sequence of isotype subgroups G_n each of which has length less than the length of G, then G must be σ-summable.*

Applying the corollary to a direct sum of σ-summable groups, we quickly obtain the following. Let $G = \bigoplus_{i \in I} G_i$ where G_i is σ-summable for each i, and let λ_i denote the length of G_i. Then G is σ-summable if and only if the length λ of G is cofinal with ω_0. In other words, G is σ-summable if and only if $\sup\{\lambda_i\}$ is cofinal with ω_0.

The next result shows that literally any reduced abelian p-group can appear as a subgroup of a σ-summable group. Hence, the condition that a reduced p-group H is a subgroup of a σ-summable group places no further restriction at all on H. In particular, H need not be σ-summable.

Proposition 2. *Any reduced abelian p-group H is a direct summand of a σ-summable group. Moreover, the complementary summand can be taken to be totally projective.*

Proof. Let the length of H be λ, which is unrestricted. Let T_n be a reduced totally projective group of length $\lambda + n$. Define T to be the direct sum of the groups T_n, and set $G = H \oplus T$. Observe that if we let $G_n = H \oplus (T_1 \oplus \cdots \oplus T_n)$, the G_n's obviously form an ascending sequence of subgroups of G having length less than G, and their union is G. Moreover, G_n being a direct summand of G is certainly isotype in G. Therefore, by the Corollary, G must be summable.

We now consider the following question posed in [D].

Question. *If G and G' are two σ-summable p-groups and $G[p]$ and $G'[p]$ are isometric, must G and G' be isomorphic?*

Answer. As it turns out, a negative answer was essentially already provided to this question by an example that appeared in these *Proceedings* more than twenty-five years ago. Indeed, there were two different papers that contained such examples, [H] and [C]. The example in the first reference is slightly better suited for our purpose here. The example to which we refer is a summable C_Ω-group that is not totally projective. For the particular example constructed in [H], the α-th Ulm invariant is $\aleph_1$ for each countable α; otherwise, it is zero. Denote this example by H.

Let H' be a direct sum of countable groups having the same Ulm invariants as H. Since H is summable, by definition, $H[p]$ is a free valuated vector space (when endowed with heights computed in H). The same, of course, is true for $H'[p]$. Therefore, H and H' have isometric socles, but neither is σ-summable since their lengths are not cofinal with ω_0. However, in the spirit of Proposition 2, this defect (in terms of a counterexample) can easily be remedied. Indeed, if we let $G = H \oplus T$ and $G' = H' \oplus T$, where T is a totally projective group of length $\omega_1 + \omega_0$, then G and G' are σ-summable (cf. Proposition 2). Moreover, the isometry between $H[p]$ and $H'[p]$ can be extended to one between $G[p]$ and $G'[p]$ by taking it to be the identity on T. Finally, the argument that G and G' are not isomorphic is simple: G' is totally projective since both H' and T are, but G cannot be totally projective since its direct summand H is not.

In conclusion, we consider now another question posed in [D]. Actually, in the interest of brevity, we only consider here the fundamental case of a more general version of the problem stated in [D].

Problem. Let F denote a perfect field of characteristic p and let G be an abelian p-group. If G is σ-summable, must $V(G)/G$ be totally projective (and hence G a direct factor of $V(G)$), where $V(G)$ denotes the group of normalized units of the group algebra $F(G)$?

(Partial) Solution. The same problem for a general p-group G, without the σ-summable hypothesis, is a well-known open problem. For a partial solution, among other sources, see [HU]. One could interpret Proposition 2 as a strong hint that σ-summability has little, if anything, to contribute as far as this problem is concerned. To make this idea more precise, we end with the following proposition. We remind the reader that we are working over a perfect field of characteristic p.

Proposition 3. *$V(G)/G$ is totally projective for all σ-summable p-groups G if and only if it is totally projective for all abelian p-groups G.*

Proof. Assume that $V(G)/G$ is totally projective for all σ-summable p-groups G. Let H be an arbitrary abelian p-group. We want to show that $V(H)/H$ is totally projective. In this regard, there is no loss of generality in assuming that H is reduced (see, for example, [M]). In accordance with Proposition 2, we can choose a totally projective p-group T so that $G = H \oplus T$ is σ-summable. If we use the multiplicative notation (as is the custom for group algebras), this becomes $G = H \times T$. By hypothesis, $V(G)/G$ is totally projective since G is σ-summable. However, it is well known that $V(G) = V(H) \times K(T)$, where $K(T)$ is the kernel of the natural map from $V(G)$ onto $V(H)$ induced by the projection of $G = H \times T$ onto its direct factor H. Since we have that $V(G)/G = (V(H)/H) \times K(T)/T$, it follows that $V(H)/H$ is totally projective since it is a direct factor of the totally projective group $V(G)/G$, which is the desired result.

References

[C] D. Cutler, *Another summable C_Ω-group*, Proc. Amer. Math. Soc. **26** (1970), 43–44.

[D] P. Danchev, *Commutative Group Algebras of a σ-Summable Abelian Group*, Proc. Amer. Math. Soc. **125** (1997), 2559–2564. MR **97k:**20011

[LM] R. Linton and C. Megibben, *Extensions of Totally Projective Groups*, Proc. Amer. Math. Soc. **64** (1977), 35–38.

[H] P. Hill, *A summable C_Ω-group*, Proc. Amer. Math. Soc. **23** (1969), 428–430.

[HU] P. Hill and W. Ullery, *A note on a theorem of May concerning commutative group algebras*, Proc. Amer. Math. Soc. **110** (1990), 59–63.

[M] W. May, *Modular group algebras of simply presented abelian groups*, Proc. Amer. Math. Soc. **104** (1988), 403–409.

Department of Mathematics, Auburn University, Alabama 36849

E-mail address: hillpad@mail.auburn.edu