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TRIANGULAR EXTENSION SPECTRUM OF WEIGHTED SHIFTS

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Abstract. A necessary and sufficient condition for a complex number to be in the triangular extension spectrum of a weighted backward shift is obtained. It is shown that the triangular extension spectrum of a weighted backward shift is always a closed annulus when it is not empty. Moreover, for any given closed annulus, there exists a weighted backward shift with the annulus as its triangular extension spectrum.

A bounded operator T acting on a separable Hilbert space H is called *triangular* if there exists an orthonormal basis $\{e_n\}_1^\infty$ for H such that Te_n is in the span of $\{e_1, e_2, \ldots, e_n\}$ for each n. A vector x is called an *algebraic vector* for an operator T if there exists a nonzero polynomial $p(z)$ such that $p(T)x = 0$. Let $\mathcal{E}_T$ be the set of algebraic vectors for an operator T. It can be shown that T is triangular if and only if $\mathcal{E}_T$ is dense in H. T is called *algebraic* if there exists a nonzero polynomial $p(z)$ so that $p(T) = 0$. An operator A is called *semitriangular* if A is an extension of a triangular operator by a finite rank operator, or equivalently, the norm closure of the set of algebraic vectors has finite codimension in H. The finite codimension is called the *index of semitriangularity* of A. Semitriangular operators have been proved to be very useful in constructing counter-examples. Several longstanding open questions were answered negatively in [W], which led to the study of semitriangular operators; see [HLP], [HLW], [LW1], [LW2].

Let $B(H)$ be the set of all bounded linear operators on H, $\sigma(T)$ denote the spectrum of T in $B(H)$, $\sigma_e(T)$ denote the essential spectrum of T, and let $\mathbb{N}$ be the set of natural numbers. For any bounded sequence of complex numbers $\{s_n\}_1^\infty$, we define the corresponding weighted backward shift S with respect to $\{e_n\}_1^\infty$ such that $Se_1 = 0$, $Se_2 = s_1e_1$, $Se_3 = s_2e_2, \ldots, Se_{i+1} = s_ie_i, \ldots$. Clearly, $\|S\| = \sup\{|s_i|\}$. Throughout this note, we always let $\mathbb{N}_0 = \{i \in \mathbb{N} : s_i = 0\}$.

The triangular extension spectrum was first introduced in [HLP] and it was shown that the triangular extension spectrum plays a very important role in studying semitriangular operators and algebraic extensions of triangular operators.

Definition 1. Let $T \in B(H)$ be a triangular operator. The triangular extension spectrum of T, denoted by $\sigma_\Delta(T)$, is defined to be the set of all $t \in \mathbb{C}$ such that

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there exists a $b \in B(\mathbb{C}, H)$ $(= H)$ with the property that

$$A = \begin{pmatrix} T & b \\ 0 & t \end{pmatrix}$$

has index of semitriangularity 1 in $H \oplus \mathbb{C}$. Let $\rho_\Delta(T)$ denote the complement of $\sigma_\Delta(T)$ in $\mathbb{C}$.

Lemma 2 ([HLP])**.** *If T is a triangular operator, then $t \in \sigma_\Delta(T)$ if and only if $\mathcal{E}_T + \operatorname{Ran}(T - tI) \neq H$.*

It follows from the above lemma that $\sigma_\Delta(T) \subseteq \sigma(T)$. In fact, it is shown that $\sigma_\Delta(T)$ is always a compact subset of $\sigma_e(T)$ ([HLP]). Theorem 2.7 of [HLW] implies $\sigma_\Delta(T) = \varnothing$ if and only if T is algebraic.

Lemma 3 ([HLW])**.** *If $\mathcal{E}_T + \operatorname{Ran}(T - tI) = H$, then there exists a positive integer k such that $\ker(T - tI)^k + \operatorname{Ran}(T - tI) = H$.*

Lemma 4. *Let S be a weighted backward shift. If S is not algebraic, then*

$$0 \in \sigma_\Delta(S) \text{ if and only if } \varliminf_{i\to\infty} |s_i| = 0.$$

Proof. "$\Leftarrow$" *Case* 1: $\mathbb{N}_0$ *is infinite.*

Suppose $\mathbb{N}_0 = \{n_1, n_2, \dots\}$ such that $n_1 < n_2 < n_3 < \cdots$. It is easy to see that S is algebraic if and only if S is nilpotent if and only if $\mathbb{N}_0$ is infinite and $\sup\{n_{j+1} - n_j\}$ is finite. Let $H_0 = [e_{n_1}, e_{n_2}, \dots, e_{n_i}, \dots]$, where $[\cdot]$ denotes the closed linear span. Since $Se_{n_i+1} = s_{n_i} e_{n_i} = 0$, $\operatorname{Ran}(S) \subseteq H_0^\perp$. Since S is not nilpotent, thus $\forall k \in \mathbb{N}$, $\exists m_k$ such that $S^k e_{m_k} \neq 0$. Suppose $n_{j_k - 1} < m_k \leq n_{j_k}$; then $S^k e_{n_{j_k}} \neq 0$. Let $b = \sum_{i=1}^\infty \frac{1}{i} e_{n_i}$. We show that $b \notin \ker(S^k) + \operatorname{Ran}(S)$ for any k, which, by Lemma 2 and Lemma 3, implies $0 \in \sigma_\Delta(S)$.

Suppose $b = u + Sv$, where $u \in \operatorname{Ker}(S^k)$. Let $u = \sum_{i=1}^\infty u_i e_i$.

Since $\operatorname{Ran}(S) \subseteq H_0^\perp$, we have $u_{n_i} = \frac{1}{i}$.

Since $S^k u = \sum_{i=1}^\infty u_i S^k e_i = 0$ and $\langle S^k e_i, S^k e_j \rangle = 0$, for all $i \neq j$, where $\langle \cdot \rangle$ denotes the inner product, we have $u_i S^k e_i = 0$, for all i, in particular, $u_{n_i} S^k e_{n_i} = 0$, i.e. $\frac{1}{i} S^k e_{n_i} = 0$, thus $S^k e_{n_i} = 0$, for all i, which is a contradiction.

Case 2: $\mathbb{N}_0$ *is finite.*

Let P_n be the orthogonal projections of H onto $[e_1, e_2, \dots, e_n]$ for $n = 1, 2, 3, \dots$. We need to show that $\ker(S^k) + \operatorname{Ran}(S) \neq H$. Since $\ker(S^i) \subseteq \ker(S^{i+1})$ for all i, we only need to show that $\ker(S^k) + \operatorname{Ran}(S) \neq H$ for k large enough. Without loss of generality, we assume k is large enough so that $s_i \neq 0$ for all $i > k$. For any $x = \sum_{i=1}^\infty x_i e_i \in \ker(S)$, $Sx = \sum_{i=1}^\infty x_i s_i e_i = 0$. Since $\mathbb{N}_0$ is finite, we have all but finitely many x_i's are zero. Therefore there exists an N such that $\ker(S) \subseteq P_N H$, so $\ker(S^k) \subseteq P_M H$, where $M = N + k$. If $\ker(S^k) + \operatorname{Ran}(S) = H$, then $\operatorname{Ran}(P_M^\perp S) = P_M^\perp \operatorname{Ran}(S) = P_M^\perp H$. Note that $P_M^\perp S = P_M^\perp S P_{M+1}^\perp$ and $s_i \neq 0$, for $i > M$. Hence $P_M^\perp S P_{M+1}^\perp$ is a 1-1 and onto map from $P_{M+1}^\perp H$ to $P_M^\perp H$. Therefore $P_M^\perp S P_{M+1}^\perp$ is invertible by the open mapping theorem, but since $\varliminf_{i\to\infty} |s_i| = 0$, there exists a subsequence s_{n_j} such that $s_{n_j} \to 0$ as $j \to \infty$. Thus $P_M^\perp S P_{M+1}^\perp e_{n_j+1} = s_{n_j} e_{n_j} \to 0$ as $j \to \infty$, contradicting the fact that $P_M^\perp S P_{M+1}^\perp$ is invertible.

"$\Rightarrow$" If $\varliminf_{i\to\infty} |S_i| \neq 0$, then there exist an $\varepsilon_0 > 0$ and a $k_0 \in \mathbb{N}$ such that $|s_i| > \varepsilon_0$, for all $i > k_0$. For any $x \in P_{k_0}^\perp H$, $x = \sum_{i=k_0+1}^\infty x_i e_i$, let $y = \sum_{i=k_0+1}^\infty \frac{1}{s_i} x_i e_{i+1}$; then $Sy = x$, i.e. $\operatorname{Ran}(S) \supseteq P_{k_0}^\perp H$. Since $P_{k_0} H \subseteq \ker(S^k)$ for k large enough, we have

$\ker(S^k) + \mathrm{Ran}(S) = H$ for k large enough. Therefore, by Lemma 2 and Lemma 3, $0 \notin \sigma_\Delta(S)$. □

Theorem 5. *Let S be a weighted backward shift. If $\mathbb{N}_0$ is infinite and S is not algebraic, then $\sigma_\Delta(S) = \sigma(S)$.*

Proof. We only need to show that $\sigma(S) \subseteq \sigma_\Delta(S)$. Let $t \in \sigma(S)$. If $t = 0$, then $t \in \sigma_\Delta(S)$ by Lemma 4. If $t \neq 0$, then $\ker(S - tI) = 0$. By Lemma 2 and Lemma 3, $t \in \sigma_\Delta(S)$ if and only if $\mathrm{Ran}(S - tI) \neq H$, which is equivalent to $t \in \sigma(S)$ by the open mapping theorem, since $\ker(S - tI) = 0$. □

Lemma 6 ([S])**.** *If S is a weighted backward shift with spectral radius r, then $\sigma(S) = \{t \colon |t| \leq r\}$.*

The following is essentially Theorem 8 of [S].

Lemma 7. *Let S be a weighted backward shift. Suppose that $\mathbb{N}_0$ is finite with $k_0 = \max\{i \colon i \in \mathbb{N}_0\}$. If $t \neq 0$, then $S - tI$ is not 1-1 if and only if*

$$(*) \qquad \sum_{n=k_0+1}^{\infty} \left| \frac{t^{n-k_0}}{s_n \cdots s_{k_0+1}} \right|^2 < \infty.$$

In this case, $\dim \ker(S - tI) = 1$.

Lemma 8. *If $t \in \sigma(S)$ and $t \neq 0$, then $t \in \rho_\Delta(S) \Leftrightarrow \mathrm{Ran}(S - tI) = H$.*

Proof. From Lemma 7, $\dim \ker(S - tI)$ is at most one, so for any k, $\dim \ker(S - tI)^k$ is finite (in fact at most k). For any natural number n and $y \in [e_1, \ldots, e_n]$, we can solve $(S - tI)x = y$ for x. Thus $\mathrm{Ran}(S - tI)$ is dense in H. Combining the above with Lemma 2 and Lemma 3, we have $t \in \rho_\Delta(S) \Leftrightarrow \exists k, \ni \ker(S - tI)^k + \mathrm{Ran}(S - tI) = H \Leftrightarrow \mathrm{Ran}(S - tI) = H$. □

Corollary 9. *If S is a weighted backward shift and S is not algebraic, then $\sigma_\Delta(S) = \sigma_e(S)$.*

Proof. This follows immediately from Lemma 4 and Lemma 8. □

Lemma 10 ([HLP])**.** *Suppose that $A \in B(H)$ has a closed range and let P be the orthogonal projection of H onto* $\mathrm{Ran}(A)$*. Then there exists an $\varepsilon > 0$ such that* $\mathrm{Ran}(A) \subseteq \mathrm{Ran}(A + \lambda P)$ *for all λ with $|\lambda| < \varepsilon$.*

Corollary 11. *If $\mathrm{Ran}(S - tI) = H$, then there exists an $\varepsilon > 0$ such that $\mathrm{Ran}(S - zI) = H$ for all z with $|z - t| < \varepsilon$.*

Lemma 12. *Suppose that $\mathbb{N}_0$ is finite with $k_0 = \max\{i \colon i \in \mathbb{N}_0\}$. If $t \in \sigma(S)$ and $t \neq 0$, then $t \in \rho_\Delta(S)$ if and only if there exist an M and a z_0 with $|z_0| > |t|$ such that*

$$(**) \qquad \left| \frac{z_0^n}{s_{k+n} s_{k+n-1} \cdots s_k} \right| \leq M, \quad \forall k > k_0,\ n > 0.$$

Proof. "$\Rightarrow$" If $t \in \rho_\Delta(S)$, and $t \neq 0$, then by Lemma 8, $\mathrm{Ran}(S - tI) = H$. Since $t \in \sigma(S)$ and $S - tI$ is surjective, $t \notin \partial\sigma(S)$. Combining with Corollary 11, we can find a disk (centered at t) contained in $\sigma(S)$ with the property that $\mathrm{Ran}(S - zI) = H$ for all z in the disk. Since the disk is contained in the spectrum, we have $S - zI$ is

not 1-1 for all z in the disk. In particular, there exists a z_0 with $|z_0| > |t|$ so that $\mathrm{Ran}(S - z_0 I) = H$ and $S - z_0 I$ is not 1-1. By Lemma 7,

$$\sum_{n=0}^{\infty} \frac{|z_0^{2n}|}{|s_{k_0+n+1} \cdots s_{k_0+1}|^2} < \infty.$$

Hence $S - z_0 I$ is onto and 1-1 from $\ker(S - z_0 I)^{\perp}$ to H, so it has a bounded inverse B from H to $\ker(S - z_0 I)^{\perp}$.

For any $k > k_0$, let $x^{(k)} = Be_k$, write $x^{(k)} = \sum_{i=1}^{\infty} x_i^{(k)} e_i$; then $\|x^{(k)}\| \le \|B\|$, in particular, $|x_i^{(k)}| \le \|B\|$ for all i. Let

$$M = \left\{ 1 + \left(\sum_{n=0}^{\infty} \frac{|z_0^{2n}|}{|s_{k_0+n+1} \cdots s_{k_0+1}|^2} \right)^{1/2} \right\} \|B\|.$$

Since $(S - z_0 I)x^{(k)} = (S - z_0 I)Be_k = e_k$, we have

$$s_i x_{i+1}^{(k)} - z_0 x_i^{(k)} = 0, \quad i \ne k, \qquad \text{and} \qquad s_k x_{k+1}^{(k)} - z_0 x_k^{(k)} = 1.$$

From the above, we can obtain

$$x_k^{(k)} = \frac{z_0}{s_{k-1}} x_{k-1}^{(k)} = \cdots = \frac{z_0^{k-k_0-1}}{s_{k-1} \cdots s_{k_0+1}} x_{k_0+1}^{(k)}.$$

Now for any positive integer n,

$$\begin{aligned} x_{k+n+1}^{(k)} &= \frac{z_0}{s_{k+n}} x_{k+n}^{(k)} = \cdots = \frac{z_0^n}{s_{k+n} \cdots s_{k+1}} x_{k+1}^{(k)} \\ &= \frac{z_0^n}{s_{k+n} \cdots s_{k+1}} \left(\frac{z_0}{s_k} x_k^{(k)} + \frac{1}{s_k} \right) = \frac{z_0^n}{s_{k+n} \cdots s_k} x_k^{(k)} + \frac{z_0^n}{s_{k+n} \cdots s_k} \\ &= \frac{z_0^n}{s_{k+n} \cdots s_k} \left(\frac{z_0^{k-k_0-1}}{s_{k-1} \cdots s_{k_0+1}} x_{k_0+1}^{(k)} \right) + \frac{z_0^n}{s_{k+n} \cdots s_k} \\ &= \frac{z_0^{n+k-k_0-1}}{s_{k+n} \cdots s_{k_0+1}} x_{k_0+1}^{(k)} + \frac{z_0^n}{s_{k+n} \cdots s_k}. \end{aligned}$$

Thus

$$\frac{z_0^n}{s_{k+n} \cdots s_k} = x_{k+n+1}^{(k)} - \frac{z_0^{n+k-k_0-1}}{s_{k+n} \cdots s_{k_0+1}} x_{k_0+1}^{(k)}.$$

Therefore

$$\left| \frac{z_0^n}{s_{k+n} \cdots s_k} \right| \le |x_{k+n+1}^{(k)}| + \left| \frac{z_0^{n+k-k_0-1}}{s_{k+n} \cdots s_{k_0+1}} \right| |x_{k_0+1}^{(k)}| \le M.$$

"$\Leftarrow$" Note that when $n = 1$, $(**)$ implies $|\frac{z_0}{s_{k+1} s_k}| \le M$, $\forall k > k_0$, so

$$\left| \frac{1}{s_k} \right| \le M \left| \frac{s_{k+1}}{z_0} \right| \le M \frac{\|S\|}{|z_0|}, \quad \forall k > k_0.$$

Let $H_1 = [e_{k_0+1},\ e_{k_0+2}, \ldots]$. We only need to show $\mathrm{Ran}(S - tI) \supseteq H_1$.

First, we define a bounded linear operator R on H_1 such that $Re_i = \frac{1}{s_i} e_{i+1}$, $\forall i \ge k_0 + 1$. It follows that $\|R\| \le M$ and $SR = I_{H_1}$. Since $(z_0 R)^n e_i = \frac{z_0^n}{s_{i+n-1} \cdots s_i} e_{i+n}$, we have $\|(z_0 R)^n\| \le \sup\{|\frac{z_0^n}{s_{i+n-1} \cdots s_i}| \colon i \ge k_0 + 1\} \le M|z_0|$, which implies $\|(tR)^n\| =$

$|\frac{t}{z_0}|^n\|(z_0R)^n\| \le |\frac{t}{z_0}|^n M|z_0|$. Therefore, $Q \equiv \sum_{n=1}^{\infty}(tR)^n$ defines a bounded linear operator on H_1. Given any $y \in H_1$, let $x = Qy$; then

$$\begin{aligned}(S-tI)x = (S-tI)Qy &= (S-tI)\sum_{n=1}^{\infty}(tR)^n y \\ &= \left(S\sum_{n=1}^{\infty}(tR)^n - t\sum_{n=1}^{\infty}(tR)^n\right)y \\ &= \left(StR + S\sum_{n=2}^{\infty}(tR)^n - t\sum_{n=1}^{\infty}(tR)^n\right)y \\ &= \left(tI_{H_1} + tI_{H_1}\sum_{n=1}^{\infty}(tR)^n - t\sum_{n=1}^{\infty}(tR)^n\right)y = ty.\end{aligned}$$

This shows $\mathrm{Ran}(S-tI) \supseteq H_1$. □

Remark. (i) $(**)$ implies $(*)$. This is because if $(**)$ holds, then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{|t^{2n}|}{|s_{k_0+n+1}\cdots s_{k_0+1}|^2} &= \sum_{n=0}^{\infty}\left(\left|\frac{t}{z_0}\right|^{2n}\frac{|z_0^{2n}|}{|s_{k+0+n+1}\cdots s_{k_0+1}|^2}\right) \\ &\le \sum_{n=0}^{\infty}\left(\left|\frac{t}{z_0}\right|^{2n}M^2\right) < \infty.\end{aligned}$$

(ii) $(**)$ is equivalent to the following: For any fixed $N > k_0$, there exist an M and a z_0 with $|z_0| > |t|$ such that

$$\left|\frac{z_0^n}{s_{k+n}s_{k+n-1}\cdots s_k}\right| \le M, \quad \forall k > N,\ n > 0.$$

Similarly, $(*)$ is equivalent to: For any $N > k_0$,

$$\sum_{n=N}^{\infty}\left|\frac{t^{n-N+1}}{s_n\cdots s_N}\right|^2 < \infty.$$

(iii) In the proof of Lemma 12, we see that $(**)$ implies $\left|\frac{1}{s_i}\right| \le M\frac{\|S\|}{|z_0|}$, $\forall i > k_0$, i.e. $\varliminf_{i\to\infty}|s_i| \ne 0$.

If $\mathbb{N}_0$ is finite, we define $r_1 \equiv \inf\{|t|\colon t \text{ does not satisfy } (**)\}$, $r_2 \equiv \inf\{|t|\colon t \text{ does not satisfy } (*)\}$. (In the case $\mathbb{N}_0$ is infinite, we define $r_1 = r_2 = 0$.)

Since $(**)$ implies $(*)$, we have $r_1 \le r_2$.

Theorem 13. *If S is a weighted backward shift and S is not algebraic, then $\sigma_\Delta(S) = \{t\colon r_1 \le |t| \le r\}$. In particular, if S is the standard backward shift, then $\sigma_\Delta(S) = \{t\colon |t| = 1\}$.*

Proof. This follows directly from Theorem 5, Lemma 6 and Lemma 12. □

Lemma 12 together with Lemma 7 implies the next corollary which gives a more descriptive picture of the spectrum of a weighted backward shift.

Corollary 14. *With assumptions as above, we have the following*:

(i) *If $|t| < r_1$, then $S - tI$ is onto but not* 1-1.
(ii) *If $r_1 \le |t| < r_2$, then $S - tI$ is neither* 1-1 *nor onto.*

(iii) *If* $r_2 < |t| \leq r$, *then* $S - tI$ *is* 1-1 *but not onto.*
(iv) *If* $|t| < r$, *then* $S - tI$ *is invertible.*

Remark. $S - r_2 I$ can be 1-1 in some cases and not 1-1 in others. In the case $r_1 = r_2$, we do not have (ii) of the above theorem and in the case $r_2 = r$, we do not have (iii) of the above theorem. However, we always have if $r_1 \leq |t| \leq r$, then $S - tI$ is not onto, that is, $\sigma_\Delta(S) = \{t \colon r_1 \leq |t| \leq r\}$ if S is not algebraic.

The following example shows we can alway choose a weighted backward shift with desired r_1, r_2 and r. For simplicity, we assume $r = 1$.

Example. Suppose that $0 < a < b < 1$. Defined a weighted backward shift as follows: For $k = 0, 1, 2 \ldots$, define

$$\begin{aligned}
s_i &= 1 \quad \frac{1}{2}k(k+1)(n+2) < i \leq \frac{1}{2}k(k+1)(n+2) + (k+1),\\
s_i &= a \quad \frac{1}{2}k(k+1)(n+2) + (k+1) < i \leq \frac{1}{2}k(k+1)(n+2) + 2(k+1),\\
s_i &= d \quad \frac{1}{2}k(k+1)(n+2) + 2(k+1) < i \leq \frac{1}{2}k(k+1)(n+2) + (n+2)(k+1)
\end{aligned}$$

where n is fixed such that $(\frac{a}{b})^{n+1} \leq b$ and $b^{n+1} \leq \frac{a}{b}$, and $d = \sqrt[n]{\frac{b^{n+2}}{a}}$. It can be verified that $r_1 = a$, $r_2 = b$ and $r = 1$.

Corollary 15. *For any* $L_1 < L_2$, *there exists a weighted backward shift* S *such that* $\sigma_\Delta(S) = \{t \colon L_1 \leq |t| \leq L_2\}$.

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References

[HLP] D. Han, D. Larson and Z. Pan, *The triangular extension spectrum and algebraic extensions of operators*, preprint.

[HLW] D. Herrero, D. Larson and W. Wogen, *Semitriangular operators*, Houston J. Math. **17** (1991), 477–499. MR **92m:**47037

[LW1] D. Larson and W. Wogen, *Extensions of bitriangular operators*, Integ. Eq. and Oper. Th. **25** (1996), 216–223. MR **97d:**47027

[LW2] ———, *Some problems on triangular and semitriangular operators*, Contemporary Mathematics **120** (1991), 97–100. MR **92i:**47019

[LW3] ———, *Reflexivity of* $T \oplus 0$, J. Funct. Anal. **92** (1990), 448–467. MR **91i:**47010

[LW4] ———, *Extension of normal operators*, Integ. Eq. and Oper. Th. **20** (1994), 325–334. MR **96b:**47024

[S] A. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Math Surveys, no. 13, Amer. Math. Soc., Providence, R.I., 1974, pp. 49–128. MR **50:**14341

[W] W. Wogen, *Some counterexamples in non-self-adjoint algebras*, Ann. of Math. **126** (1987), 415–427. MR **89b:**47066

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