PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 11, November 1998, Pages 3335–3336
S 0002-9939(98)04715-7

ON THE CENTER CONDITIONS OF CERTAIN CUBIC SYSTEMS

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(Communicated by Hal L. Smith)

ABSTRACT. This paper provides a new simple proof of a recent result by C. B. Collins (Differential and Integral Equations **10** (1997), 333–356) to derive the center conditions for a class of planar cubic systems. The idea is to consider periodic solutions of a related scalar non-autonomous equation.

Consider the planar system

$$\begin{aligned} \dot{x} &= -y + x(\alpha x + \beta y + Ax^2 + Bxy + Cy^2), \\ \dot{y} &= x + y(\alpha x + \beta y + Ax^2 + Bxy + Cy^2), \end{aligned} \tag{1}$$

where α, β, A, B, and C are real constants. In this paper, we give a simple and short proof to the following recent result of [2].

Theorem. *The origin is a center for* (1) *if and only if*

$$A + C = 0 \text{ and } A\alpha^2 + B\alpha\beta + C\beta^2 = 0.$$

Proof. The system (1) in polar coordinates r and θ becomes

$$\dot{r} = p(\theta)r^2 + q(\theta)r^3,$$
$$\dot{\theta} = 1,$$

with $p(\theta) = \alpha\cos\theta + \beta\sin\theta$ and $q(\theta) = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$.

The origin is a center for (1) if and only if every solution in a neighborhood of $r = 0$ is a $2\pi-$periodic solution for the differential equation

$$\frac{dr}{d\theta} = p(\theta)r^2 + q(\theta)r^3. \tag{2}$$

Let $r(\theta, c)$ be the solution of (2) with $r(0, c) = c$. For small c, we write

$$r(\theta, c) = \sum_{n=1}^{\infty} a_n(\theta)c^n, \tag{3}$$

where $a_1(0) = 1$ and $a_n(0) = 0$ for $n > 1$. The origin is a center if and only if $a_1(2\pi) = 1$, and $a_n(2\pi) = 0$ for all $n > 1$. Substituting (3) into (2) and equating the coefficients of c yield

$$\dot{a}_n = p \sum_{i+j=n} a_i a_j + q \sum_{i+j+k=n} a_i a_j a_k. \tag{4}$$

Received by the editors April 1, 1997.

1991 *Mathematics Subject Classification.* Primary 34C25; Secondary 34C05, 34C23.

Solving these equations recursively gives

$$a_1 = 1, a_2 = \overline{p}, a_3 = \overline{p}^2 + \overline{q}, a_4 = \overline{p}^3 + 2\overline{p}\,\overline{q} + \overline{\overline{p}q},$$

and

$$a_5 = \overline{p}^4 + 3\overline{p}^2\overline{q} + \overline{\overline{p}^2 q} + 2\overline{p}\,\overline{\overline{p}q} + \frac{3}{2}\overline{q}^2,$$

where a bar over a function denotes its indefinite integral. The functions a_2 and a_4 are integrals of polynomials of odd degrees; hence, they are $2\pi-$periodic. Therefore, the first two necessary conditions for a center are $a_3(2\pi) = 0$ and $a_5(2\pi) = 0$. We have

$$a_3(2\pi) = \pi(A + C).$$

If $A + C = 0$, then

$$a_5 = -\frac{\pi}{2}(A\alpha^2 + B\alpha\beta - A\beta^2).$$

Now, we prove that these conditions are also sufficient. If $\alpha\beta \neq 0$, then straightforward manipulations imply that $q = \frac{-A}{\alpha\beta}p\overline{p}$. It follows inductively from (4) that $\dot{a}_n = p\,F(\overline{p})$, for some continuous function F. Therefore, $a_n(2\pi) = \overline{F}(\overline{p}(2\pi)) - \overline{F}(\overline{p}(0)) = 0$. If $\alpha = A = 0$ and $\beta \neq 0$, then $q = -\frac{B}{\beta}p\overline{p}$; when $\beta = A = 0$ and $\alpha \neq 0$, then $q = -\frac{B}{\alpha}p\overline{p}$. In the case $\alpha = \beta = 0$, the equation has the form $\frac{dr}{d\theta} = q(\theta)r^3$. Since q has mean value zero, the solutions of this equation are $2\pi-$periodic.

The idea of this proof is the same one used in [1] for systems with homogeneous nonlinearities.

References

[1] M.A.M. Alwash and N.G. Lloyd, *Non-autonomous equations related to polynomial two-dimensional systems*, Proc. Royal Soc. Edinburgh, 105A(1987), 129-152. MR **88i:**58145

[2] C.B. Collins, *Conditions for a center in a simple class of cubic system*, Differential and Integral Equations, 10(2)(1997), 333-356. CMP 97:05

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