**Qqpi GROUPS AND QUASI-EQUIVALENCE**

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**Abstract.** A torsion-free abelian group $G$ is *qpi* if every map from a pure subgroup $K$ of $G$ into $G$ lifts to an endomorphism of $G$. The class of *qpi* groups has been extensively studied, resulting in a number of nice characterizations. We obtain some characterizations for the class of homogeneous *Qqpi* groups, those homogeneous groups $G$ such that, for $K$ pure in $G$, every $\theta : K \to G$ has a lifting to a quasi-endomorphism of $G$. An irreducible group is *Qqpi* if and only if every pure subgroup of each of its strongly indecomposable quasi-summands is strongly indecomposable. A *Qqpi* group $G$ is *qpi* if and only if every endomorphism of $G$ is an integral multiple of an automorphism. A group $G$ has minimal test for quasi-equivalence (*mtqe*) if whenever $K$ and $L$ are quasi-isomorphic pure subgroups of $G$ then $K$ and $L$ are equivalent via a quasi-automorphism of $G$. For homogeneous groups, we show that in almost all cases the *Qqpi* and *mtqe* properties coincide.

All groups considered in the paper will be torsion-free abelian of finite rank. We assume familiarity with the standard tools used in studying these groups, such as types, pure subgroups, $p$-rank and quasi-isomorphism. We use $\equiv$ for quasi-equality and $H^n$ for a direct sum of $n$ copies of a group $H$. For $W$ a (torsion-free abelian) group or ring, we write $QW$ for the divisible hull of $W$. We start by defining the class of groups that will be the object of our attention.

**Definition 1.** A group $G$ is *Qqpi* if whenever $0 \to K \to G$ is pure exact then the induced sequence $QE(G) \to QHom(K, G) \to 0$ is exact.

What we are saying, simply, is that for any homomorphism $\theta$ mapping a pure subgroup $K$ into $G$ there is a $\phi \in QE(G)$ such that $\phi |_K = \theta$. Thus, there is a positive integer $t$ such that $t\theta$ can be lifted to an endomorphism of $G$.

Plainly any *qpi* group (where we require that $\theta$ itself be lifted to an endomorphism of $G$) is *Qqpi*. For a complete characterization of the class of *qpi* groups see [A-O’B-R] together with [R-2]. We provide examples to show that the *Qqpi* groups form a considerably larger class.

It follows directly from the definitions that a homogeneous *Qqpi* group $G$ must be irreducible, that is, $QE(G)$ will act transitively on the rank one subspaces of $QG$. The precise relation between homogeneous *Qqpi* groups and irreducible groups is given in our first theorem.

**Theorem 2.** Let $G$ be irreducible and write $G \cong H^n$, where $H$ is strongly indecomposable and irreducible ([R-1], Th. 5.5). Then: (a) $H$ is *Qqpi* if and only if...
each pure subgroup of \( H \) is strongly indecomposable, and \( (b) \) \( G \) is \( \text{Qqpi} \) if and only if \( H \) is \( \text{Qqpi} \).

Proof. \( (a) \) Since \( H \) is irreducible strongly indecomposable then \( \text{QE}(H) = \Gamma \), a division ring (\([\text{R}-1]\)). Thus, if \( K \) is pure in \( H \) with a nontrivial quasi-decomposition \( K \cong U \oplus V \) then \( i \oplus 0 : K \to H \) (\( i : U \to H \) is inclusion) cannot have a lifting to an element of \( \text{QE}(H) \). It follows that if \( H \) is \( \text{Qqpi} \) then every pure subgroup of \( H \) must be strongly indecomposable. Conversely, assume that every pure subgroup of \( H \) is strongly indecomposable. Suppose that \( K \subset H \) is pure and \( \theta : K \to H \). Let \( X \) be a rank one pure subgroup of \( K \). Since \( H \) is irreducible, \( \theta \mid_X \) has a lifting to \( \phi \in \text{QE}(H) \). If \( X = K \) we’re done. Otherwise we show that \( \phi \mid_K = \theta \) by showing that \( \phi \mid_L = \theta \mid_L \) for each rank two pure subgroup \( L \subset K \) with \( X \subset L \). To see this, note that, since \( L \) is homogeneous strongly indecomposable, then by Baer’s Lemma (\([\text{A}], \text{Lemma 1.12}\)) type \( L/X > \text{type } L = \text{type } H \). Hence \( \text{Hom}(L/X, H) = 0 \). It follows that \( (\phi - \theta) \mid_L = 0 \) and the proof of \( (a) \) is complete.

\( (b) \) It is simple to check that the class of \( \text{Qqpi} \) groups is closed under taking quasi-summands. Hence, if \( G \) as above is \( \text{Qqpi} \) then so is \( H \).

Conversely, assume that \( H \) is \( \text{Qqpi} \) and let \( \theta : K \to G \) with \( K \) pure in \( G \). We show that \( \theta \) can be lifted to a quasi-endomorphism of \( G \) by induction on rank \( K \). Since \( G \) is irreducible, \( \theta \) can be quasi-lifted when rank \( K = 1 \). Suppose that rank \( K > 1 \) and that, for any pure subgroup \( L \subset G \) of smaller rank, any homomorphism from \( L \) to \( G \) can be lifted to an element of \( \text{QE}(G) \). Let \( L \subset K \) be a pure subgroup with rank \( K/L = 1 \). By the inductive assumption there exists \( \phi \in \text{QE}(G) \) such that \( \phi \mid_L = \theta \mid_L \). Let \( \psi = (\phi \mid_K - \theta) : K \to G \). If \( \psi = 0 \) we’re done. If \( \psi \neq 0 \), since \( K \) is homogeneous of the same type as \( G \), we can apply Baer’s Lemma to obtain a quasi-decomposition \( K = L \oplus X \) with \( X \) a rank one pure subgroup of \( G \).

Recall that \( \Gamma = \text{QE}(H) \) can be identified with the centralizer of the simple \( \text{QE}(G) \) module \( QG \) (\([\text{R}-1]\)). We claim that we cannot have \( \Gamma X \subset \Gamma L \) (as \( \Gamma \)-subspaces of \( QG \)). To prove the claim suppose \( 0 \neq x \in \Gamma X \) can be written \( x = \gamma l \) for some \( \gamma \in \Gamma, l \in L \). Let \( QG = \Gamma l \oplus Y \), where \( Y \) is any complementary \( \Gamma \)-subspace of \( QG \), and let \( \rho \) be the associated projection of \( QG \) onto \( \Gamma l \). Then \( \rho \in \text{Hom}_\Gamma(QG, QG) = \text{QE}(G) \), the equality by the double centralizer theorem. Using \( \rho \) we obtain a quasi-decomposition \( G = G' \oplus G'' \) with \( G' = \Gamma l \cap G \). Note that \( G' \) is quasi-isomorphic to \( H \), since \( G' \) is a quasi-summand of \( G \) with rank \( G' = \text{rank } \Gamma l \cap G = \text{rank } \Gamma = \text{rank } H \). But, since \( L \oplus X \subset \Gamma \), a pure subgroup of \( G \), the pure subgroup of \( G' \) generated by \( x \) and \( l \) will be rank two completely decomposable, contradicting part \( (a) \).

Thus \( \Gamma X \cap \Gamma L = 0 \). Arguing as in the previous paragraph, we obtain a quasi-decomposition \( G = G' \oplus G'' \) with \( L \subset G', X \subset G'' \). Let \( \phi' \in \text{QE}(G) \) be such that \( \phi' \mid_X = \theta \mid_X \). Then \( \phi \mid_L \oplus \phi' \mid_{G''} \) will be the desired quasi-lifting of \( \theta \).

Example 3. Let \( F \) be an algebraic number field of dimension 3 over \( Q \) such that in its ring of integers \( J \) there is a decomposition of an integral prime \( p, pJ = PP' \), with \( \dim_{K/pZ}(J/P) = 2 \). (Such examples are easy to construct, e.g. see \([\text{A-O-B-R}]\).) Let \( R = Jp \). Since \( R \) is a full subring of the field \( F; R \) will be irreducible as an additive group. Furthermore, since rank \( R = 3 \) and \( p \text{-rank } R = 2 \), in the quasi-decomposition \( R = H^n \) we must have \( n = 1, R = H \). Thus \( R \) is strongly indecomposable. Note that \( qR = R \) for all integral primes \( q \neq p \), so that \( ZqJ \subset R \). Denote \( Jp = ZpJ \). Since \( R \) has \( p \text{-rank two, } R/Jp \cong Z(p^\infty) \). Let \( K \) be a rank two pure subgroup of \( R \). Since \( R \) is strongly indecomposable we cannot have \( K + Jp = R \).
Hence \((K + J_p)/J_p\) is finite. It follows that any rank two pure subgroup of \(R\) must be isomorphic to \(Z_p \oplus Z_p\). By Theorem 2 (a), \(R\) is not \(Q_{qpi}\).

We prove for future reference that the additive group \(R\) satisfies the following property, which is weaker than the \(Q_{qpi}\) property: if \(U, V\) are quasi-isomorphic pure subgroups of \(R\) (which by our above discussion just means that rank \(U = \text{rank } V\)) then \(\phi U \cong V\) for \(\phi\) a quasi-automorphism of \(R\) (which in this case is simply a nonzero element of \(QE(R) \cong F\)). Since \(R\) is irreducible, if \(U, V\) are rank one pure subgroups of \(R\), then \(\phi U \cong V\) for \(\phi \in QE(R)\). Write \(F = Q(r)\) for \(r \in R\) and let \(U_0 = Z_p1 \oplus Z_pr\). We show that for each pure subgroup \(V \subset R\) of rank two there exists \(\phi \in QE(R)\) with \(\phi U_0 \cong V\). The fact that arbitrary rank two pure subgroups \(U, V \subset R\) are equivalent via a quasi-automorphism of \(R\) will follow immediately. By rank considerations \(rv \cap V \neq 0\). Take \(0 \neq v \in V\) such that \(rv \in V\). Then, if \(\phi\) is multiplication by \(v\), we have \(\phi U_0 \cong V\).

The characterization of homogeneous strongly indecomposable \(qpi\) groups \(G\) from [A-O'B-R] and [R-2] is that \(G \cong R \otimes A\), where rank \(A = 1\) and \(R = E(G)\) is a strongly homogeneous ring of \(p\)-rank one for some integral prime \(p\). A strongly homogeneous ring is a full subring \(R\) of an algebraic number field such that every element of \(R\) is an integral multiple of a unit \(U\). Since the class of \(Q_{qpi}\) groups is closed under quasi-isomorphism and the class of groups of the form \(R \otimes A\) as above is not, we have some immediate simple examples of strongly indecomposable homogeneous \(Q_{qpi}\) groups which fail to be \(qpi\). We also note, in connection with Theorem 2 (b), that [A-O'B-R] has an example of a strongly indecomposable homogeneous \(qpi\) group \(H\) such that \(H \oplus H\) is not \(qpi\).

Our \(G\) constructed in the next example shows that the endomorphism ring of a strongly indecomposable homogeneous \(Q_{qpi}\) group need not be a strongly homogeneous ring and need not have \(p\)-rank one for any prime \(p\).

**Example 4.** There is a rank 4 homogeneous \(Q_{qpi}\) group \(G\) such that \(E(G)\) is the ring of integral quaternions.

**Construction of the example.** Let \(R\) denote the ring of integral quaternions and let \(S = \{0 \neq s = ai + bj + ck \in R \mid \text{the first nonzero coefficient of } s\text{ is positive and } \gcd(a, b, c) = 1\}\). Enumerate the elements of \(S\), say \(S = \{s_m\}\). For \(s_m = a_m i + b_m j + c_m k\), put \(N(s_m) = a_m^2 + b_m^2 + c_m^2\). Let \(P_m\) be the set of integral primes \(p\) such that \(-N(s_m)\) is a nonzero square mod \(p\). Since each \(-N(s_m) \neq 0\), it is well known that each \(P_m\) will be an infinite set of primes. For each \(m\), choose an infinite subset \(P'_m \subset P_m\) so that \(\{P'_m \mid 1 \leq m < \infty\}\) will be a collection of disjoint sets. For each \(p \in P'_m\) choose \(d_p \in Z\) with \(0 < d_p < p\) such that \(-N(s_m) \equiv d_p^2 \mod p\). Let \(G\) be the \(R\)-submodule of \(QR\) generated by \(R\) and \(\{(d_p - s_m)/p \mid 1 \leq m < \infty, p \in P'_m\}\).

Plainly \(R \subset E(G)\), so that \(QR \subset QE(G)\). Since \(QR\) is a division algebra and \(QR = QG\) it follows that \(G\) is irreducible, hence homogeneous. We show that type \(G = 0\) (= type \(Z\)) by considering the divisibility of the element \(1 \in G\). Clearly, \(1\) is divisible in \(G\) by no prime in the complement of \(\bigcup_m P'_m\). Suppose that \(1 = pg\) for \(g \in G, p \in P'_m\). Writing \(g\) as a finite \(R\)-combination of the generators of \(G\) produces the equation \(1 = r'(d_p - s_m) + pr'\) for some \(r, r' \in R\). Multiplying this equation by \((d_p + s_m)\) yields the equation \(d_p + s_m = r[d_p^2 + N(s_m)] + pr'(d_p + s_m)\). Since \(d_p^2 \equiv -N(s_m) \mod p\) we have \((d_p + s_m) \in pR\), so that \(d_p\) is divisible by \(p\), a contradiction. Hence, for all primes \(p, 1 \not\in pG\). Thus type \(G = \text{type}_G(1) = 0\).
Suppose that $K \subset G$ is a pure subgroup with a nontrivial quasi-decomposition $K = U \oplus V$. Since $G/R$ is torsion, $U \cap R$ and $V \cap R$ are both nonzero. Take nonzero elements $u = (e + ai + bj + ck) \in U \cap R$, $v \in V \cap R$ and denote $\bar{u} = e - ai - bj - ck$. The element $\bar{u}v \in R$ can be written in the form $t + ls_m$ for some integers $t, l$ and positive integer $m$. It follows that, for each $p \in P'_m$, $[(ld_p + t) - \bar{u}v] = l(d_p - s_m) \in pG$. Multiplication by $u$ yields that $[((ld_p + t)u - N(u)v) \in pG$ for all $p \in P'_m$. This is impossible, since $N(u) \neq 0$ and $u, v$ are elements lying in different quasi-summands of a pure subgroup of $G$, a homogeneous group of type $0$. The resulting contradiction shows that each pure subgroup of $G$ is strongly indecomposable. By Theorem 2 (a), $G$ is $Qqpi$.

Since the quaternion algebra $QR$ is contained in $QE(G)$ and $QR = QG, G$ is irreducible. Moreover, $G$ itself is strongly indecomposable, so that rank $QE(G) = \text{rank } QG$. Hence, $QR = QE(G)$. We have already noted that $R \subset E(G)$, so $R \subset E(G) \subset QR$. By considering the action of a possible endomorphism of $G$ on the set of generators for $G$, it is easy to check that $E(G)$ coincides precisely with $R$.

If $G$ is strongly indecomposable, homogeneous and $qpi$, then $QE(G)$ is a field. Since quasi-isomorphic groups have isomorphic quasi-endomorphism rings, our example additionally shows that the class of $Qqpi$ groups is larger than the class of groups quasi-isomorphic to a $qpi$ group. The following simple result gives the precise connection between the $Qqpi$ and $qpi$ properties for strongly indecomposable homogeneous groups.

**Theorem 5.** Let $H$ be strongly indecomposable homogeneous $Qqpi$. Then $H$ is $qpi$ if and only if every element of $R = E(H)$ is an integral multiple of a unit of $R$.

**Proof.** Let $H$ be strongly indecomposable homogeneous $Qqpi$. By Theorem 2 (a) every pure subgroup of $H$ is strongly indecomposable. By Theorem B of [A-O’B-R], to show that $H$ is $qpi$ it suffices to show that for each pair of rank one pure subgroups $X, Y$ of $H$ there exists a unit $u \in R$ with $uX = Y$. Since $H$ is homogeneous $Qqpi$, there exists $0 \neq r \in R$ with $rX \subset Y$. If $r = tu, t \in Z, u$ a unit of $R$, it follows that $uX = Y$.

**Definition 6.** A group $G$ has minimal test for quasi-equivalence (mtqe) if whenever $U$ and $V$ are quasi-isomorphic pure subgroups of $G$ then there exists $\phi$, a quasi-automorphism of $G$, with $\phi U \cong V$.

We have already noted that the group in Example 3 is homogeneous with mtqe but is not $Qqpi$. For a second example, if we eliminate the element $s_1$ from our construction of Example 4, then we obtain a subgroup $G' \subset G$ in which $\langle 1 \rangle \oplus \langle s_1 \rangle$ is pure. However, it is not too hard to show that $G'$ remains strongly indecomposable and that $E(G')$ will coincide with $E(G)$, the ring of integral quaternions. Thus, $G'$ is a strongly indecomposable homogeneous non-$Qqpi$ group. It also is not too hard to show that $G'$ has mtqe. Since the construction of this second example is not central to our work, we omit the details.

Note that the group $G$ of Example 3 has $QE(G)$ a number field of degree 3. The group $G$ in the modification of Example 4 would have $QE(G)$ a division algebra of degree 2. The next result, which we feel is somewhat surprising, shows that only division algebras of degree 2 or 3 can occur as $QE(G)$ for a strongly indecomposable homogeneous group $G$ for which the $Qqpi$ and mtqe properties fail to coincide.

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Theorem 7. Let $G$ be irreducible and write $G \cong H^n$ with $H$ strongly indecomposable and irreducible. Suppose that $\dim_Q F > 3$, where $F$ is a maximal subfield of the division algebra $\Gamma = QE(H)$. Then $G$ is $Qqpi$ if and only if $G$ has $mtqe$.

Proof. First we assume that $n = 1$, that is, $H = G$ is strongly indecomposable. Since $\Gamma = QE(H)$, every $0 \neq r \in E(H)$ is a quasi-automorphism. Hence, if $H$ is $Qqpi$, it follows immediately that $H$ has $mtqe$. Conversely, suppose that $H$ has $mtqe$. By Theorem 2 (a), to prove that $H$ is $Qqpi$ we need to show that any pure subgroup of $H$ is strongly indecomposable. It will be enough to show that any rank two pure subgroup of $H$ is strongly indecomposable.

Let $K$ be a rank two pure subgroup of $H$; say $K = \langle x, y \rangle$, the pure subgroup generated by elements $x, y$. Since $H$ is irreducible there exists $\gamma \in QE(H)$ with $\gamma x = y$. Extend the field $Q(\gamma)$ to a maximal subfield $F \subset QE(H)$.

Suppose that for each primitive element $\alpha \in E(H)$ with $F = Q(\alpha)$ the pure subgroup $\langle x, \alpha x \rangle < H$ is completely decomposable. By the proof of the existence of a primitive element for algebraic number fields, for $\alpha$ primitive we can choose a positive integer $t$ such that $\beta = \alpha + ta^2$ will also be primitive. Then the pure subgroups $\langle x, \alpha x \rangle$ and $\langle x, \beta x \rangle$ will be isomorphic (both being completely decomposable subgroups of the homogeneous group $H$). Because $H$ has $mtqe$ we have $\phi(x, \alpha x) = \langle x, \beta x \rangle$, for some $\phi \in \Gamma$. Thus $\phi(x) = (q_1 + q_2 \beta)x$ for some rational numbers $q_1, q_2$. Since $\Gamma$ is a division algebra, $\phi = (q_1 + q_2 \beta)$. But $\phi(\alpha x) = (q_1 + q_2 \beta)\alpha x$ cannot be a rational combination $q_3 x + q_4 \beta x$, for then $(q_1 + q_2 \beta)\alpha = q_3 + q_4 \beta$ for some rationals $q_3, q_4$. Since $\beta = \alpha + ta^2$, this latter equation would contradict the fact that $\alpha$ is algebraic over $Q$ of degree greater than three. It follows that, for some primitive element $\alpha \in E(H)$, the pure subgroup $\langle x, \alpha x \rangle$ will be strongly indecomposable.

As before, the type of the rank one factor group $\langle x, \alpha x \rangle / \langle x \rangle$ must be greater than the type of $H$. Thus, one of two possibilities must occur.

Case I: There is an infinite set of primes $P$ such that for $p \in P$ there exists an integer $c_p$ with $h_p(\alpha x - c_p x) > h_p(x)$. Here $h_p$ denotes the $p$-height of an element in $H$. In this case an easy calculation shows that, for every integer $t$ with $1 \leq t < \deg \alpha$ and $p \in P$, we have $h(\alpha^t x - c_p^t x) > h_p(x)$. Let $L$ be the pure subgroup of $H$ generated by $x$ and $\{\alpha^t x \mid 1 \leq t < \deg \alpha\}$. Our set of height inequalities shows that the inner type of $L / \langle x \rangle_*$ is greater than the type of $H$. Since $y = \gamma x \in L$ and $K = \langle x, y \rangle$, then $\langle x \rangle_*$ of $L / \langle x \rangle$ is a pure subgroup of $L / \langle x \rangle_*$, hence type $[K / \langle x \rangle_*]$ > type $H$, so that $K$ cannot be homogeneous and completely decomposable. Thus $K$ must be strongly indecomposable, as desired.

Case II: For some prime $p$ with $pH \neq H$ there is a set of integers $c_n, 1 \leq n < \infty$, with $\langle \alpha x - c_n x \rangle \in p^n H$. Arguing as in Case I, we can conclude that $L / \langle x \rangle_*$ is $p$-divisible; hence so is $K / \langle x \rangle_*$. Again we have that $K$ cannot be homogeneous and completely decomposable, so $K$ is strongly indecomposable. The proof that $H$ is $Qqpi$ is complete.

Now suppose that $n > 1$ and $G \cong H^n$ is as in the statement of the theorem. If $G$ has $mtqe$ it is immediate that $H$ has $mtqe$. By what we have proved already, $H$ is $Qqpi$ and, by Theorem 2 (b), $G$ is $Qqpi$. Conversely, let $G$ be $Qqpi$. Then $H$ is also $Qqpi$, so, by the remark at the beginning of the proof, $H$ has $mtqe$. In view of the fact that $G \cong H^n$ is irreducible with $QE(G) \cong (\Gamma)^n$, it is not hard to see that, for $X,Y$ any rank one pure subgroups of $G$, we can choose a quasi-automorphism
\( \phi \in QE(G) \) with \( \phi X = Y \). With this observation, the proof of Theorem 2 (b) goes through *mutatis mutandis* to show that \( G \) has mtqe.

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**References**


