

## QUASINORMAL SUBRELATIONS OF ERGODIC EQUIVALENCE RELATIONS

ALEXANDRE I. DANILENKO

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**ABSTRACT.** We introduce a notion of quasinormality for a nested pair  $\mathcal{S} \subset \mathcal{R}$  of ergodic discrete hyperfinite equivalence relations of type  $II_1$ . (This is a natural extension of the normality concept due to Feldman-Sutherland-Zimmer.) Such pairs are characterized by an irreducible pair  $F \subset Q$  of countable amenable groups or rather (some special) their Polish closure  $\overline{F} \subset \overline{Q}$ . We show that “most” of the ergodic subrelations of  $\mathcal{R}$  are quasinormal and classify them. An example of a nonquasinormal subrelation is given. We prove as an auxiliary statement that two cocycles of  $\mathcal{R}$  with dense ranges in a Polish group are weakly equivalent.

### 0. INTRODUCTION

It is well known that two ergodic finite measure-preserving actions of countable amenable groups are orbit equivalent [Dy], [CFW]. This can be rephrased in equivalent terms of measured equivalence relations [FM]: there exists a unique (up to isomorphism) hyperfinite discrete ergodic equivalence relation, say  $\mathcal{R}$ , of type  $II_1$ . A natural subsequent problem that arises here is to study subrelations of  $\mathcal{R}$  and this is the main concern of the present paper.

It was shown in [FSZ] how to associate a countable index set  $J$  and a cocycle  $\sigma : \mathcal{R} \rightarrow \Sigma(J)$  to any pair  $\mathcal{S} \subset \mathcal{R}$  of discrete ergodic type  $II_1$  equivalence relations, where  $\Sigma(J)$  is the full permutation group of  $J$ . The cardinality of  $J$  is called the *index* of  $\mathcal{S} \subset \mathcal{R}$  and is related closely to the Jones index in the study of sub-von-Neumann-algebras [Jo]. The cocycle  $\sigma$  is called the *index cocycle* of  $\mathcal{S} \subset \mathcal{R}$ . The weak equivalence class of  $\sigma$  depends only on the isomorphism class of the pair  $\mathcal{S} \subset \mathcal{R}$ .

J. Feldman, C. E. Sutherland and R. J. Zimmer provided an elegant classification of ergodic hyperfinite pairs  $\mathcal{S} \subset \mathcal{R}$  in the following two cases: (a)  $\mathcal{S}$  is normal, (b)  $\mathcal{S}$  is of finite index in  $\mathcal{R}$  [FSZ]. Remark that the case (b) was considered earlier by M. Gerber in a different context—she classified the finite extensions of ergodic probability-preserving transformations up to the “factor orbit equivalence” [Ge]. The purpose of this paper is to extend the above results to a wider class of subrelations, namely quasinormal ones.

We call  $\mathcal{S}$  *quasinormal* if  $\sigma$  (or its restriction to  $\mathcal{S}$ ) is regular, i.e.  $\sigma$  is cohomologous to a cocycle with dense range in a closed subgroup of  $\Sigma(J)$ . The concept of

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quasinormality was introduced in a different way in a previous paper of the author [Da, §4], where the problem of genericity for extensions of  $\mathcal{S}$ -cocycles to  $\mathcal{R}$ -cocycles with values in amenable locally compact groups was discussed (see also [GLS]). We show that the above definition is equivalent to [Da, Definition 4.1].

Before proceeding with the statements of our main results, we recall some standard orbit theory notation. Let  $\mathcal{P}$  be a discrete measured equivalence relation on a standard probability space  $(X, \mathfrak{B}, \mu)$ . By the full group  $[P]$  we mean the group of automorphisms of  $X$  whose orbits are contained in  $\mathcal{P}$ -classes. The normalizer  $N[\mathcal{R}]$  of  $[P]$  is the group of automorphisms of  $X$  which preserve  $\mathcal{P}$  (see §2 for rigorous definitions). Two  $\mathcal{R}$ -subrelations  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are  $\mathcal{R}$ -conjugate if  $\mathcal{S}_1 = (T \times T)\mathcal{S}_2$  for a transformation  $T \in N[\mathcal{R}]$ .

We say that a pair  $F \subset Q$  of Polish groups is *irreducible* if  $F$  contains no nontrivial closed normal subgroups of  $Q$ .

**Theorem 0.1 (Canonical Form for Quasinormal Subrelations).** *Let  $\mathcal{S}$  be an ergodic quasinormal subrelation of  $\mathcal{R}$ . There exist an ergodic subrelation  $\mathcal{P} \subset \mathcal{S}$ , a countable amenable group  $Q \subset N[\mathcal{P}]$ , and a subgroup  $F$  of  $Q$  such that  $Q \cap [P] = \{\text{Id}\}$ ,  $F \subset Q$  is irreducible,  $\mathcal{R}$  is generated by  $\mathcal{P}$  and  $Q$  and  $\mathcal{S}$  is generated by  $\mathcal{P}$  and  $F$ . Moreover, the index cocycle may be realized as  $\sigma = \rho \circ \theta : \mathcal{R} \rightarrow \Sigma(F \backslash G)$ , where  $\theta : \mathcal{R} \rightarrow Q$  is given by  $\theta(x, y) = q$  for all  $(x, y) \in \mathcal{P}$  and  $q \in Q$ , and  $\rho$  is the Cayley representation of  $Q$  in  $\Sigma(F \backslash Q)$  as right translations.*

Notice that the pair  $F \subset Q$  is not determined uniquely (up to isomorphism) by  $\mathcal{S}$ . That is why we need to introduce some special equivalence relation for these objects as follows. Denote by  $\overline{Q}$  (resp.  $\overline{F}$ ) the closure of  $\rho(Q)$  (resp. the closure of  $\rho(F)$ ) in  $\Sigma(F \backslash Q)$  endowed with the usual Polish topology. It is easy to see that  $\overline{F} = \{q \in \overline{Q} \mid q(F) = F\}$ . Hence  $\overline{F}$  is an open subgroup of  $\overline{Q}$  and  $\overline{F} \subset \overline{Q}$  is an irreducible pair of Polish groups.

**Definition 0.2.** We say that two irreducible pairs of countable groups  $F_1 \subset Q_1$  and  $F_2 \subset Q_2$  are weakly isomorphic if there exists a continuous isomorphism of  $\overline{Q}_1$  onto  $\overline{Q}_2$  which takes  $\overline{F}_1$  onto  $\overline{F}_2$ .

**Theorem 0.3 (Classification of Quasinormal Subrelations).** *There is a bijective correspondence between the ergodic quasinormal subrelations  $\mathcal{S}$  of  $\mathcal{R}$  (up to  $\mathcal{R}$ -conjugacy) and the weak isomorphism classes of irreducible pairs of countable amenable groups  $F \subset Q$ . Furthermore,  $F \subset Q$  is related to  $\mathcal{S}$  as described in Theorem 0.1.*

Notice that the normal subrelations are quasinormal—they correspond exactly to the case where  $F$  is trivial. Clearly, the subrelations of finite index are also quasinormal, since the index cocycle as well as every cocycle with values in a finite group is regular. In both cases  $\overline{Q} = Q$ ,  $\overline{F} = F$  and Theorem 0.3 gives [FSZ, Theorems 3.1, 3.2].

The outline of the paper is as follows. §1 is of a preliminary nature. We study cocycles of  $\mathcal{R}$  with values in Polish groups and extend some results from [GS], where the groups were assumed to be locally compact. In particular, we prove that two cocycles with dense ranges in a Polish group are weakly equivalent. The second section introduces the idea of a quasinormal pair  $\mathcal{S} \subset \mathcal{R}$  (cf. [Da, §4]). The proofs of Theorems 0.1 and 0.3 and related problems are contained here. In the final §3, we show that a “typical” (in the Baire category sense) ergodic subrelation of  $\mathcal{R}$

is quasinormal but nonnormal. We also provide an example of a nonquasinormal subrelation.

Remark that throughout this paper equivalence relations are of type  $II_1$ . However, all the results are also valid for type  $II_\infty$  equivalence relations with minor modifications of the arguments. We hope to treat the type  $III$  case in a later paper.

# 1. COCYCLES OF MEASURED EQUIVALENCE RELATIONS WITH VALUES IN POLISH GROUPS

We begin this section with some background on orbit theory. Let  $(X, \mathfrak{B}, \mu)$  be a standard probability space. Denote by  $\text{Aut}(X, \mathfrak{B}, \mu)$  the group of its automorphisms, i.e. Borel, one-to-one, onto,  $\mu$ -preserving transformations; we do not distinguish between two of them which agree on a  $\mu$ -conull subset. Let  $\mathcal{R} \subset X \times X$  be a Borel discrete (i.e. each equivalence class is countable) equivalence relation. We shall assume that  $\mathcal{R}$  is  $\mu$ -preserving, i.e. there exists a countable subgroup  $\Gamma \subset \text{Aut}(X, \mu)$  such that  $\mathcal{R}$  is the  $\Gamma$ -orbital equivalence relation. We endow  $\mathcal{R}$  with the induced Borel structure and the  $\sigma$ -finite measure  $\mu_{\mathcal{R}}$ ,  $d\mu_{\mathcal{R}}(x, y) = d\mu(x)$ ,  $(x, y) \in \mathcal{R}$ . Write also

$$[\mathcal{R}] = \{q \in \text{Aut}(X, \mu) \mid (qx, x) \in \mathcal{R} \text{ for } \mu\text{-a.a. } x \in X\},$$

$$N[\mathcal{R}] = \{q \in \text{Aut}(X, \mu) \mid (qx, qy) \in \mathcal{R} \text{ for } \mu_{\mathcal{R}}\text{-a.a. } (x, y) \in \mathcal{R}\}$$

for the *full group* of  $\mathcal{R}$  and the *normalizer* of  $[\mathcal{R}]$  respectively.  $\mathcal{R}$  is called *hyperfinite* if it can be generated by a single automorphism.

Let  $G$  be a Polish group with  $1_G$  the identity of  $G$ . A Borel map  $\alpha : \mathcal{R} \rightarrow G$  is a (1-)cocycle of  $\mathcal{R}$  if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \quad \text{for a.a. } (x, y), (y, z) \in \mathcal{R}.$$

We do not distinguish between two cocycles if they agree  $\mu_{\mathcal{R}}$ -a.e. Two cocycles,  $\alpha, \beta : \mathcal{R} \rightarrow G$ , are *cohomologous* ( $\alpha \approx \beta$ ), if

$$\alpha(x, y) = \phi(x)^{-1}\beta(x, y)\phi(y) \quad \text{for } \mu_{\mathcal{R}}\text{-a.a. } (x, y),$$

where  $\phi : X \rightarrow G$  is a Borel function (we call it a *transfer function* from  $\alpha$  to  $\beta$ ). A cocycle is a *coboundary* if it is cohomologous to the trivial one.

Two cocycles  $\alpha, \beta : \mathcal{R} \rightarrow G$  are *weakly equivalent* if there is a transformation  $T \in N[\mathcal{R}]$  such that  $\alpha \approx \beta \circ T$ , where the cocycle  $\beta : \mathcal{R} \rightarrow G$  is defined by  $\beta \circ T(x, z) = \beta(Tx, Tz)$ .

We assume from now on that  $\mathcal{R}$  is ergodic, i.e. every  $\mathcal{R}$ -saturated Borel subset is  $\mu$ -null or  $\mu$ -conull.

We say that  $\alpha$  has *dense range* in  $G$  if for every  $A \in \mathfrak{B}$ ,  $\mu(A) > 0$ , and an open subset  $O \subset G$  there exists  $B \in \mathfrak{B}$  and a transformation  $q \in [\mathcal{R}]$  with  $\mu(B) > 0$ ,  $B \cup qB \subset A$ , and  $\alpha(x, qx) \in O$  for all  $x \in B$ .

**Proposition 1.1.** *Let  $F, H$  be closed subgroups of  $G$ , and let two cocycles  $\alpha, \beta : \mathcal{R} \rightarrow G$  take values and have dense ranges in  $F$  and  $H$  respectively. If  $\alpha \approx \beta$ , then  $F$  and  $H$  are conjugate in  $G$ .*

*Proof.* Let  $\alpha(x, y) = \phi(x)^{-1}\beta(x, y)\phi(y)$  at  $\mu_{\mathcal{R}}$ -a.e.  $(x, y) \in \mathcal{R}$  for a Borel function  $\phi : X \rightarrow G$ . Take any proper value  $g_0 \in G$  of  $\phi$ , which means that  $\mu(\phi^{-1}(O)) > 0$  for every neighborhood  $O$  of  $g_0$ . We shall prove that  $F = g_0^{-1}Hg_0$ . Given any  $g \in H$  and a neighborhood  $V$  of  $g_0^{-1}gg_0$ , we choose neighborhoods  $U$  of  $g_0$  and

$W$  of  $g$  with  $U^{-1}WU \subset V$ . Since  $\beta$  has dense range in  $H$ , there exists a Borel subset  $A \subset X$  and a transformation  $q \in [\mathcal{R}]$  such that  $\mu(A) > 0$ ,  $A \cup qA \subset \phi^{-1}(U)$  and  $\beta(x, qx) \in W$  for all  $x \in A$ . Recall that  $\alpha(\mathcal{R}) \subset F$  and hence  $V \cap F \neq \emptyset$ . Since  $V$  is an arbitrary neighborhood of  $g_0^{-1}gg_0$ , we deduce that  $g_0^{-1}gg_0 \in F$ . Thus  $g_0^{-1}Hg_0 \subset F$ . The converse inclusion is established in a similar way.  $\square$

*Remark 1.2.* It is easy to deduce from the above proof that the transfer function  $\phi$  is of the form  $\phi(x) = \psi(x)g'$  a.e. for some  $g' \in G$  and a Borel function  $\psi : X \rightarrow N_G(H)$ , where  $N_G(H) := \{g \in G \mid gHg^{-1} = H\}$  is the normalizer of  $H$  in  $G$ .

**Definition 1.3.** A cocycle  $\alpha : \mathcal{R} \rightarrow G$  is called *regular* if it is cohomologous to a cocycle which takes values and has dense range in a closed subgroup  $H$  of  $G$ .

We denote by  $\langle \alpha \rangle$  the conjugacy class of  $H$ , i.e.  $\langle \alpha \rangle = \{gHg^{-1} \mid g \in G\}$ . It is well defined by Proposition 1.1. It is obvious that given a cocycle  $\alpha$  with dense range in  $G$ , then  $\alpha \circ T$  also has dense range in  $G$  for every transformation  $T \in N[\mathcal{R}]$ . We deduce from this fact and Proposition 1.1

**Corollary 1.4.** *Let  $\alpha$  and  $\beta$  be weakly equivalent cocycles. If  $\alpha$  is regular, then so is  $\beta$  and  $\langle \alpha \rangle = \langle \beta \rangle$ .*

Recall that an equivalence relation  $\mathcal{P}$  is of *type I* if there is a Borel subset  $A \subset X$ ,  $\mu(A) > 0$ , such that for a.e.  $x \in X$  there is a unique  $y \in A$  with  $(x, y) \in \mathcal{P}$ . We call such  $A$  a  $\mathcal{P}$ -*fundamental domain*. It is well known that every cocycle of an equivalence relation of type I is a coboundary [FM].

**Lemma 1.5** (cf. [GS, Proposition 1.1]). *Let  $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$  for an increasing sequence of type I equivalence relations  $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \dots$ . Given two cocycles  $\alpha, \beta : \mathcal{R} \rightarrow G$ , consider two sequences of Borel maps  $a_n, b_n : X \rightarrow G$  such that  $\alpha(x, y) = a_n(x)a_n(y)^{-1}$ ,  $\beta(x, y) = b_n(x)b_n(y)^{-1}$  for a.e.  $(x, y) \in \mathcal{R}_n$ . Define a sequence of maps  $f_n : X \rightarrow G$  by setting  $f_n(x) = a_n(x)b_n(x)^{-1}$ . If  $f_n$  converges a.e. to a map  $\phi : X \rightarrow G$  as  $n \rightarrow \infty$ , then  $\alpha(x, y) = \phi(x)\beta(x, y)\phi(y)^{-1}$  for a.e.  $(x, y) \in \mathcal{R}$ .*

*Proof.* For a.e.  $(x, y) \in \mathcal{R}_n$  and every  $m > n$  we have

$$\begin{aligned} f_m(x)\beta(x, y)f_m(y)^{-1} &= a_m(x)b_m(x)^{-1}b_m(x)b_m(y)^{-1}b_m(y)a_m(y)^{-1} \\ &= a_m(x)a_m(y)^{-1} = \alpha(x, y), \end{aligned}$$

since  $\mathcal{R}_n \subset \mathcal{R}_m$ . Pass to the limit to obtain  $\phi(x)\beta(x, y)\phi(y)^{-1} = \alpha(x, y)$  for a.e.  $(x, y) \in \mathcal{R}_n$ ,  $n \in \mathbb{N}$ .  $\square$

**Proposition 1.6.** *Let  $\mathcal{R}$  be hyperfinite and  $G'$  a countable dense subgroup of  $G$ . Given a cocycle  $\alpha : \mathcal{R} \rightarrow G$ , there exists a cocycle  $\beta \approx \alpha$  with  $\beta(\mathcal{R}) \subset G'$ .*

*Proof.* Since  $\mathcal{R}$  is hyperfinite, there exists an increasing sequence of type I equivalence relations  $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \dots$  with  $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$ . Let  $F_n$  stand for an  $\mathcal{R}_n$ -fundamental domain. We also put  $F_0 = X$ . Define a Borel map  $T_n : X \rightarrow F_n$  by setting  $T_n x = y$  if  $(x, y) \in \mathcal{R}_n$ . Notice that  $T_n$  is  $\mathcal{R}_n$ -invariant—i.e.  $T_n x = T_n y$  for a.e.  $(x, y) \in \mathcal{R}_n$ —and

$$(1-1) \quad \alpha(x, y) = \alpha(x, T_n x)\alpha(T_n x, T_n y)\alpha(T_n y, y) = \alpha(x, T_n x)\alpha(y, T_n y)^{-1}$$

for a.e.  $(x, y) \in \mathcal{R}_n$ . Consider the family of Borel maps  $a_n : F_{n-1} \rightarrow G$  given by  $a_n(x) = \alpha(x, T_n x)$ . Then

$$\begin{aligned} (1-2) \quad \alpha(x, T_n x) &= \alpha(x, T_1 x) \alpha(T_1 x, T_2 x) \dots \alpha(T_{n-1} x, T_n x) \\ &= a_1(x) a_2(T_1 x) \dots a_n(T_{n-1} x) \end{aligned}$$

for a.e.  $x \in X$ . Conversely, it is easy to see that an arbitrary family of Borel functions  $a_n : F_{n-1} \rightarrow G$ ,  $n \in \mathbb{N}$ , determines a cocycle  $\alpha : \mathcal{R} \rightarrow G$  by (1-1) and (1-2).

Let  $\{W_n\}_{n=1}^\infty$  be a fundamental system of neighborhoods of  $1_G$  with the following properties:  $W_n^{-1} = W_n$  and  $W_{n+1} W_{n+1} W_{n+1} \subset W_n$ ,  $n \in \mathbb{N}$ . Enumerate the elements of  $G'$ :  $G' = \{g_i\}_{i=1}^\infty$ . For each  $n \in \mathbb{N}$ , we have  $G = \bigcup_{i=1}^\infty W_n g_i$ . Hence there is an  $m_n \in \mathbb{N}$  such that  $\mu(A_n) > 1 - 2^{-n}$ , where

$$A_n := \{x \in X \mid \alpha(x, T_n x) \in \bigcup_{i=1}^{m_n} W_{n+2} g_i\}.$$

Let  $V_n$  be a neighborhood of  $1_G$  with  $g_i V_n g_i^{-1} \subset W_n$  for all  $i = 1, \dots, m_{n-2}$ ,  $n > 2$ . Take a family of Borel maps  $b_n : F_{n-1} \rightarrow G'$  such that  $a_n(x) b_n(x)^{-1} \in V_{n+1}$  for all  $x \in F_{n-1}$ . This family determines a cocycle  $\beta : \mathcal{R} \rightarrow G$ . We have for  $k \in \mathbb{N}$  and  $x \in \bigcap_{i=n}^{n+k-1} A_i$

$$\begin{aligned} f_{n+k} &:= \alpha(x, T_{n+k} x) \beta(x, T_{n+k} x)^{-1} \\ &= \alpha(x, T_{n+k-1} x) a_{n+k}(T_{n+k-1} x) b_{n+k}(T_{n+k-1} x)^{-1} \beta(x, T_{n+k-1} x)^{-1} \\ &\in \alpha(x, T_{n+k-1} x) V_{n+k+1} \beta(x, T_{n+k-1} x)^{-1} \\ &= \alpha(x, T_{n+k-1} x) V_{n+k+1} \alpha(x, T_{n+k-1} x)^{-1} \alpha(x, T_{n+k-1} x) \beta(x, T_{n+k-1} x)^{-1} \\ &\subset W_{n+k+1} W_{n+k+1} W_{n+k+1} \alpha(x, T_{n+k-1} x) \beta(x, T_{n+k-1} x)^{-1} \\ &\subset W_{n+k} \alpha(x, T_{n+k-1} x) \beta(x, T_{n+k-1} x)^{-1} \subset \dots \\ &\subset W_{n+k} W_{n+k-1} \dots W_{n+1} \alpha(x, T_n x) \beta(x, T_n x)^{-1} \subset W_n f_n(x). \end{aligned}$$

Since  $\mu(\bigcap_{i=n}^{n+k-1} A_i) \geq 1 - 2^{-n} - 2^{-n-1} - \dots - 2^{-n-k+1} \geq 1 - 2^{-n+1} \rightarrow 1$ , the sequence  $f_n$  converges in measure as  $n \rightarrow \infty$ . Hence a subsequence of  $f_n$  converges a.e. and  $\alpha \approx \beta$  by Lemma 1.5.  $\square$

*Remark 1.7.* If  $G'$  is normal in  $G$ , then the conclusion of Proposition 1.6 follows from the Connes-Krieger cohomology lemma (see [Su], [JT]). For  $G$  locally compact (and any  $G'$ ) the conclusion of the proposition was proved in [GS, Proposition 1.2]. We modified the argument of V. Ya. Golodets and S. D. Sinelshchikov in such a way to avoid the use of the local compactness.

**Proposition 1.8.** *Let  $\mathcal{R}$  be hyperfinite. Given a cocycle  $\alpha : \mathcal{R} \rightarrow G$  with dense range in  $G$ , there exists a cocycle  $\beta \approx \alpha$  such that  $\{(x, y) \in \mathcal{R} \mid \beta(x, y) = 1_G\}$  is an ergodic subrelation of  $\mathcal{R}$ .*

*Proof.* By virtue of Dye's theorem [Dy] we may assume that  $(X, \mathfrak{B}, \mu)$  and  $\mathcal{R}$  are of the following special form:

- (a)  $(X, \mu) = (\{0, 1\}, \lambda)^\mathbb{N}$ , where  $\lambda$  is the equidistribution on  $\{0, 1\}$ , i.e.  $\lambda(0) = \lambda(1) = 0.5$ ,
- (b)  $\mathcal{R} = \bigcup_{n=1}^\infty \mathcal{R}_n$ , where  $\mathcal{R}_n = \{(x, y) \in X \times X \mid x_i = y_i \text{ for all } i \geq n\}$ .

Let  $\{W_n\}_{n=1}^\infty$  be a fundamental system of neighborhoods of  $1_G$  with the properties as above. We construct inductively an increasing sequence  $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \dots$  of type  $I$  subrelations of  $\mathcal{R}$ . Describe in general the  $n$ -th step.

Let  $F_n := \{x \mid x_i = 0 \text{ for all } i < n\}$ . Clearly,  $\mu(F_{n-1} \setminus F_n) = \mu(F_n)$ . Since  $\alpha$  has dense range in  $G$ , we apply the standard exhaustion argument to construct a Borel isomorphism  $t_n : F_{n-1} \setminus F_n \rightarrow F_n$  such that  $(x, t_n x) \in \mathcal{R}$  and  $\alpha(x, t_n x) \in W_n$ . Define a Borel map  $T_n : X \rightarrow F_n$  by setting

$$T_n x = \begin{cases} x, & \text{for } x \in F_n, \\ t_n T_{n-1} T_{n-2} \dots T_1, & \text{otherwise.} \end{cases}$$

Now we put  $\mathcal{S}_n = \{(x, y) \mid T_n x = T_n y\}$ . Since the  $\mathcal{S}_n$ -class of a.e.  $x \in X$  is finite,  $\mathcal{S}_n$  is of type  $I$ . Moreover,  $F_n$  is an  $\mathcal{S}_n$ -fundamental domain. Clearly,  $\mathcal{S}_1 \subset \dots \subset \mathcal{S}_n \subset \mathcal{R}$ .

Now we put  $\mathcal{S} = \bigcup_{n=1}^\infty \mathcal{S}_n$ . Then  $\mathcal{S}$  is an ergodic subrelation of  $\mathcal{R}$ . Actually, if a Borel function  $f : X \rightarrow \mathbb{R}$  is  $\mathcal{S}_n$ -invariant, then it does not depend on the first  $n$ -coordinates of  $x$ . Since  $n$  is arbitrary,  $f$  is equal a.e. to a constant, as desired.

We claim that  $\alpha \upharpoonright \mathcal{S}$  is a coboundary. Notice that  $\alpha(x, y) = \alpha(x, T_n x) \alpha(y, T_n y)^{-1}$  for a.e.  $(x, y) \in \mathcal{S}_n$  and

$$\begin{aligned} f_{n+k} &:= \alpha(x, T_{n+k} x) = \alpha(x, T_{n+k-1} x) \alpha(T_{n+k-1} x, t_{n+k} T_{n+k-1} x) \\ &\in \alpha(x, T_{n+k-1} x) W_{n+k} \subset \dots \subset \alpha(x, T_n x) W_{n+1} W_{n+2} \dots W_{n+k} \subset f_n(x) W_n \end{aligned}$$

for a.e.  $x \in X$ . Hence  $f_n$  converges a.e. to a map  $\phi : X \rightarrow G$ . By Lemma 1.5  $\alpha(x, y) = \phi(x) \phi(y)^{-1}$  for a.e.  $(x, y) \in \mathcal{S}$ . This implies that the cocycle  $\beta(x, y) := \phi(x)^{-1} \alpha(x, y) \phi(y)$ ,  $(x, y) \in \mathcal{R}$ , satisfies the conclusion of the proposition.  $\square$

We conclude this section with an extension of the remarkable Uniqueness Theorem for Cocycles (due to V. Ya. Golodets and S. D. Sinelshchikov) to cocycles with dense ranges in Polish groups.

**Theorem 1.9.** *Let  $\alpha, \beta : \mathcal{R} \rightarrow G$  be two cocycles with dense ranges in  $G$ . If  $\mathcal{R}$  is hyperfinite then  $\alpha$  and  $\beta$  are weakly conjugate.*

*Proof.* This is almost the same as that of [GS, Lemma 1.12], where  $G$  was assumed to be locally compact, but one should use Proposition 1.6 instead of [GS, Proposition 1.2].  $\square$

## 2. QUASINORMAL SUBRELATIONS

We begin this section with a brief exposition of the basic notions of *measurable index theory* [FSZ].

Let  $\mathcal{R}$  be an ergodic  $\mu$ -preserving equivalence relation on  $(X, \mathfrak{B}, \mu)$  and  $\mathcal{S}$  an ergodic subrelation of  $\mathcal{R}$ . Then there exist  $N \in \mathbb{N} \cup \{\infty\}$  and Borel functions  $\phi_j : X \rightarrow X$  so that  $\{\mathcal{S}[\phi_j(x)] \mid 0 \leq j < N\}$  is a partition of  $\mathcal{R}[x]$ , where  $\mathcal{R}[x]$  (resp.  $\mathcal{S}[x]$ ) stands for the  $\mathcal{R}$ - (resp.  $\mathcal{S}$ -) class of  $x$ .  $N$  is called the *index* of  $\mathcal{S}$  in  $\mathcal{R}$  and  $\{\phi_j\}_j$  are called *choice functions* for the pair  $\mathcal{S} \subset \mathcal{R}$ . We may assume without loss of generality that  $\phi_j \in \text{Aut}(X, \mu)$ ,  $j \in J$ , and  $\phi_0(x) = x$  for all  $x \in X$ . Denote by  $\Sigma(J)$  the full permutation group of the set  $J \stackrel{\text{def}}{=} \{0, 1, \dots, N-1\}$  for  $N < \infty$  or  $J \stackrel{\text{def}}{=} \{0, 1, 2, \dots\}$  for  $N = \infty$ . We define the *index cocycle*  $\sigma : \mathcal{R} \rightarrow \Sigma(J)$  by setting  $\sigma(x, y)(i) = j$  if  $\mathcal{S}[\phi_i(y)] = \mathcal{S}[\phi_j(x)]$ . Notice that although choice functions are nonunique, the cohomological class of  $\sigma$  is independent of their particular choice

and is an invariant of  $\mathcal{S} \subset \mathcal{R}$ . Moreover, any cocycle cohomologous to an index cocycle arises from a suitable selection of choice functions.

Two subrelations  $\mathcal{S}_1, \mathcal{S}_2$  of  $\mathcal{R}$  are said to be  $\mathcal{R}$ -conjugate if there is a transformation  $T \in N[\mathcal{R}]$  such that  $(T \times T)\mathcal{S}_1 = \mathcal{S}_2$ . In view of [FSZ, Theorem 1.6]  $\mathcal{S}_1$  is isomorphic to  $\mathcal{S}_2$  if and only if their indices are equal and their index cocycles are weakly equivalent.

Let  $\sigma$  stand for the index cocycle of  $\mathcal{S} \subset \mathcal{R}$ . Then  $\mathcal{S}$  is said to be *normal* in  $\mathcal{R}$  if the restriction  $\sigma \upharpoonright \mathcal{S}$  of  $\sigma$  to  $\mathcal{S}$  is a coboundary. Equivalently, there are choice functions  $\{\phi_j\}_{j \in J}$  with  $\phi_j \in N[\mathcal{S}]$ ,  $j \in J$ . If, in addition,  $\mathcal{R}$  is hyperfinite, then by [FSZ, §2] there is a countable amenable group  $Q \subset N[\mathcal{S}]$  with  $Q \cap [\mathcal{S}] = 1_Q$  and such that  $\mathcal{R}$  is generated by  $\mathcal{S}$  and  $Q$ .

**Definition 2.1.**  $\mathcal{S}$  is called *quasinormal* if  $\sigma$  is regular.

From now on  $\mathcal{R}$  is an ergodic hyperfinite equivalence relation on  $(X, \mathfrak{B}, \mu)$ .

*Proof of Theorem 0.1.* By Proposition 1.8 there exists an index cocycle  $\sigma : \mathcal{R} \rightarrow \Sigma(J)$  such that the subrelation  $\mathcal{P} := \{(x, y) \in \mathcal{R} \mid \sigma(x, y) = \text{Id}\}$  is ergodic. Replacing, if necessary,  $\mathcal{S}$  by a  $\mathcal{R}$ -conjugated subrelation we may assume that  $\sigma$  is determined by a family of choice functions  $\{\phi_j\}_{j \in J}$  with the properties:  $\sigma(x, \phi_j(x))(0) = j$  for all  $x \in X$ ,  $j \in J$  and  $\mathcal{S} = \{(x, y) \in \mathcal{R} \mid \sigma(x, y)(0) = 0\}$  (see [FSZ, Theorem 1.6]). Clearly,  $\mathcal{P} \subset \mathcal{S}$  and  $\phi_j \in N[\mathcal{P}]$ ,  $j \in J$ . Let  $\{\psi_i\}_{i \in I}$  be choice functions for the pair  $\mathcal{P} \subset \mathcal{S}$ . We claim that  $\psi_i \in N[\mathcal{P}]$ . Actually, given  $(x, y) \in \mathcal{P}$ , we have

$$(\psi_i(x), \psi_i(y)) \in \mathcal{P} \iff \sigma(\psi_i(x), \psi_i(y))(j) = j \text{ for all } j \in J.$$

Since  $\sigma(\psi_i(x), \psi_i(y)) = \sigma(\psi_i(x), \phi_j \circ \psi_i(x))\sigma(\phi_j \circ \psi_i(x), \phi_j \circ \psi_i(y))\sigma(\phi_j \circ \psi_i(y), \psi_i(y))$  and  $\sigma(\phi_j \circ \psi_i(x), \phi_j \circ \psi_i(y))(0) = 0$ , we deduce that  $\sigma(\psi_i(x), \psi_i(y)) = 0$  for all  $j \in J$  and hence  $\psi_i \in N[\mathcal{P}]$ , as claimed. Notice that  $\{\psi_i \circ \phi_j\}_{i \in I, j \in J}$  are choice functions for  $\mathcal{P} \subset \mathcal{R}$ . As in [FSZ], we define a multiplication law on  $I \times J$  by setting

$$(i_1, j_1) * (i_2, j_2) = (i_3, j_3) \iff (\psi_{i_1} \circ \phi_{j_1} \circ \psi_{i_2} \circ \phi_{j_2}(x), \psi_{i_3} \circ \phi_{j_3}(x)) \in \mathcal{P} \text{ a.e.}$$

Then  $(I \times J, *)$  is a countable amenable group, say  $Q$ , and  $(I \times \{0\}, *)$  is a subgroup of  $Q$ , say  $F$  [FSZ]. Moreover, the map  $v : Q \ni q = (i, j) \mapsto \psi_i \circ \phi_j \in N[\mathcal{P}]$  is an *outer near homomorphism*, i.e. (a)  $v(q) \in [\mathcal{P}]$  if and only if  $q = 1_Q$ , (b)  $v(q_1 * q_2) \in v(q_1)v(q_2)[\mathcal{P}]$ . Since  $Q$  is amenable, there exists a map  $w : Q \rightarrow [\mathcal{P}]$  such that the map  $Q \ni q \mapsto v(q)w(q) \in N[\mathcal{P}]$  is an outer homomorphism [FSZ]. Thus  $Q$  can be viewed as a subgroup of  $N[\mathcal{P}]$ . Clearly,  $\{\phi_j w((0, j))\}_{j \in J}$  are choice functions for  $\mathcal{S} \subset \mathcal{R}$  (they determine the very same index cocycle  $\sigma$ ) and  $\{\psi_i w((i, 0))\}_{i \in I}$  are choice functions for  $\mathcal{P} \subset \mathcal{S}$ . Hence the following properties are satisfied: (a)  $Q \cap [\mathcal{P}] = \text{Id}$ , (b)  $\mathcal{R}$  is generated by  $\mathcal{P}$  and  $Q$ , (c)  $\mathcal{S}$  is generated by  $\mathcal{P}$  and  $F$ . For  $(i, j) \in Q$  and a.e.  $(x, y) \in \mathcal{P}$  we have

$$\begin{aligned} \sigma(x, \psi_i \circ \phi_j(y))(j_1) = j_2 &\iff (\phi_{j_1} \circ \psi_i \circ \phi_j(x), \phi_{j_2}(y)) \in \mathcal{S} \\ &\iff \exists i_1 \in I \text{ with } (\psi_{i_1} \circ \phi_{j_1} \circ \psi_i \circ \phi_j(x), \phi_{j_2}(y)) \in \iff (i_1, j_1) * (i, j) = (0, j_2). \end{aligned}$$

It is clear that the map  $\pi : Q \ni (i, j) \mapsto j \in J = F \backslash G$  is the  $F$ -quotient map taking  $F$  to  $\{0\}$ . Hence  $\rho((i, j))(j_1) = \pi((i_1, j_1) * (i, j)) = \pi((0, j_2)) = j_2$ . To put it another way,  $\sigma(x, qy)(j_1) = \rho(q)(j_1)$  for a.e.  $(x, y) \in \mathcal{P}$ ,  $q \in Q$ ,  $j_1 \in F \backslash G$ , i.e.  $\sigma = \rho \circ \theta$ , as desired. To complete the proof, we observe that the kernel of  $\rho$  is trivial, since  $\Sigma(J)$  acts freely on  $J$ . This implies that  $F \subset Q$  is irreducible.  $\square$

*Remark 2.2.* We observe that  $\sigma$  takes values in  $\rho(Q)$  and  $\overline{\rho(Q)} \in \langle \sigma \rangle$ . In a similar way, the restriction of  $\sigma$  to  $\mathcal{S}$  takes values in  $\rho(F)$  and  $\overline{\rho(F)} \in \langle \sigma \upharpoonright \mathcal{S} \rangle$ .

The proof of Theorem 0.3 is divided into several lemmas.

**Lemma 2.3.** *Let  $\mathcal{S}_1, \mathcal{S}_2$  be two ergodic quasinormal subrelations of  $\mathcal{R}$ . They are  $\mathcal{R}$ -conjugate if and only if their indices are equal and  $\langle \sigma_1 \rangle = \langle \sigma_2 \rangle$ , where  $\sigma_1$  and  $\sigma_2$  denote the index cocycles of  $\mathcal{S}_1$  and  $\mathcal{S}_2$  respectively.*

*Proof.* The proof follows from Theorem 1.9 and [FSZ, Theorem 1.6].  $\square$

**Lemma 2.4.** *Let  $F_1 \subset Q_1$  and  $F_2 \subset Q_2$  be two irreducible pairs of countable amenable groups corresponding to a quasinormal subrelation  $\mathcal{S}$  of  $\mathcal{R}$  as in Theorem 0.1. Then they are weakly isomorphic.*

*Proof.* Without loss of generality we may assume that  $Q_i$  is a transitive subgroup of  $\Sigma(J)$ ,  $F_i = \{q \in Q_i \mid q(0) = 0\}$ , the index cocycle  $\sigma_i$  takes values in  $Q_i \subset \Sigma(J)$  and has dense range in  $\overline{Q_i}$ , and the restriction  $\sigma_i \upharpoonright \mathcal{S}_i$  takes values in  $F_i$  and has dense range in  $\overline{F_i}$ ,  $i = 1, 2$ . Since  $\sigma_1 \approx \sigma_2$ , there is a Borel function  $\phi : X \rightarrow \Sigma(J)$  with  $\sigma_1(x, y) = \phi(x)^{-1} \sigma_2(x, y) \phi(y)$  for a.e.  $(x, y) \in \mathcal{R}$ . Let  $\tau \in \Sigma(J)$  be a proper value of  $\phi$ . By Proposition 1.1  $\overline{Q_1} = \tau^{-1} \overline{Q_2} \tau$ ,  $\overline{F_1} = \tau^{-1} \overline{F_2} \tau$  and hence the pairs  $F_1 \subset Q_1$  and  $F_2 \subset Q_2$  are weakly isomorphic.  $\square$

**Lemma 2.5.** *Let  $F \subset Q$  correspond to  $\mathcal{S}$  as in Theorem 0.1 and  $T$  be any automorphism from  $N[\mathcal{R}]$ . Then  $F \subset Q$  corresponds also to the  $\mathcal{R}$ -subrelation  $(T \times T)\mathcal{S}$ .*

**Lemma 2.6.** *For each irreducible pair of countable amenable groups  $F \subset Q$  there exists a quasinormal ergodic subrelation  $\mathcal{S} \subset \mathcal{R}$  such that  $F \subset Q$  corresponds to  $\mathcal{S}$ .*

*Proof.* It is well known that  $Q$  can be embedded into  $N[\mathcal{R}]$  in such a way that  $Q \cap [\mathcal{R}] = \text{Id}$ . Denote by  $\mathcal{R}'$  (resp.  $\mathcal{S}'$ ) the equivalence relation generated by  $\mathcal{R}$  and  $Q$  (resp. by  $\mathcal{P}$  and  $F$ ). Since  $\mathcal{R}'$  is hyperfinite, there is a transformation  $T \in \text{Aut}(X, \mu)$  with  $(T \times T)\mathcal{R}' = \mathcal{R}$ . Clearly, the subrelation  $\mathcal{S} := (T \times T)\mathcal{S}'$  is as desired.  $\square$

*Proof of Theorem 0.3.* In view of Theorem 0.1 and Lemmas 2.4–2.6 the map  $\{\mathcal{R}\text{-conjugacy class of } \mathcal{S}\} \mapsto \{\text{the weak isomorphism class of } F \subset Q \text{ as in Theorem 0.1}\}$  is well defined and onto. It remains to verify the injectivity. Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be two quasinormal subrelations of  $\mathcal{R}$  such that the corresponding pairs  $F_1 \subset Q_1$  and  $F_2 \subset Q_2$  are weakly isomorphic. Since  $\text{Card}(F_1 \setminus Q_1) = \text{Card}(F_2 \setminus Q_2)$ , the  $\mathcal{R}$ -indices of  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are equal. Let  $\sigma_1, \sigma_2 : \mathcal{R} \rightarrow \Sigma(J)$  stand for the index cocycles of  $\mathcal{S}_1$  and  $\mathcal{S}_2$  respectively. It is clear that  $\overline{Q_1}$  and  $\overline{Q_2}$  viewed as closed subgroups of  $\Sigma(J)$  are conjugate. Since  $\overline{Q_1} \in \langle \sigma_1 \rangle$  and  $\overline{Q_2} \in \langle \sigma_2 \rangle$ , it follows from Lemma 2.3 that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are  $\mathcal{R}$ -conjugate.  $\square$

Recall that  $\mathcal{S}$  is normal if  $\sigma \upharpoonright \mathcal{S}$  is a coboundary. Hence it is natural to state

**Proposition 2.7.**  *$\mathcal{S}$  is quasinormal if and only if  $\sigma \upharpoonright \mathcal{S}$  is regular.*

*Proof.* ( $\implies$ ) Without loss of generality we may assume that  $\sigma$  takes values and has dense range in a closed transitive subgroup  $G \subset \Sigma(J)$  and  $\mathcal{S} = \{(x, y) \in \mathcal{R} \mid \sigma(x, y)(0) = 0\}$ . Since  $H := \{\tau \in G \mid \tau(0) = 0\}$  is an open subgroup of  $G$  and  $\mathcal{S} = \sigma^{-1}(H)$ , it follows that  $\sigma \upharpoonright \mathcal{S}$  has dense range in  $H$ .

( $\impliedby$ ) Let  $\sigma \upharpoonright \mathcal{S}$  take values and have dense range in a closed subgroup  $H \subset \Sigma(J)$ . By Proposition 1.8 we may assume that  $\mathcal{P} := \{(x, y) \in \mathcal{S} \mid \sigma(x, y) = \text{Id}\}$  is an



ergodic subrelation. It remains to repeat the argument of Theorem 0.1 almost literally to deduce that  $\mathcal{S} \subset \mathcal{R}$  has the structure described in Theorem 0.1.  $\square$

### 3. GENERIC PROPERTIES OF SUBRELATIONS

Denote by  $Z^1$  the set of  $\mathcal{R}$ -cocycles with values in  $\Sigma(J)$ . Let  $\lambda$  be a  $\mu_{\mathcal{R}}$ -equivalent probability measure on  $\mathcal{R}$ . It is well known that  $Z^1$  endowed with the topology of convergence in  $\lambda$  is a Polish space [FM]. This topology is unaffected if we replace  $\lambda$  with an equivalent probability measure. Let  $\delta$  stand for Haar ( $\sigma$ -finite) measure on  $J$ . The group  $\text{Aut}(X \times J, \mu \times \delta)$  of  $\mu \times \delta$ -preserving automorphisms of  $X \times J$  is Polish when endowed with the weak topology. Recall that  $R_n \rightarrow R$  *weakly* in  $\text{Aut}(X \times J, \mu \times \delta)$  if  $(\mu \times \delta)(R_n A \Delta R A) + (\mu \times \delta)(R_n^{-1} A \Delta R^{-1} A) \rightarrow 0$  as  $n \rightarrow \infty$  for each Borel subset  $A \subset X \times J$  with  $(\mu \times \delta)(A) < \infty$ . By [CK] the ergodics, say  $\mathcal{E}$ , form a dense  $G_\delta$  in  $\text{Aut}(X \times J, \mu \times \delta)$ . Since  $\mathcal{R}$  is hyperfinite, there exists an ergodic transformation  $T \in \text{Aut}(X, \mu)$  such that  $\mathcal{R}$  is the  $T$ -orbital equivalence relation. Consider the map  $\Phi : Z^1 \ni \alpha \mapsto T_\alpha \in \text{Aut}(X \times J, \mu \times \delta)$ , where  $T_\alpha$  is given by  $T_\alpha(x, j) = (Tx, \alpha(x)(j))$ . It is routine to verify that  $\Phi$  is continuous. Let  $Z_{\text{ind}}^1$  stand for the set of index cocycles, i.e.

$$Z_{\text{ind}}^1 = \{\alpha \in Z^1 \mid \alpha \text{ is the index cocycle of some ergodic subrelation } \mathcal{S} \subset \mathcal{R}\}.$$

Clearly,  $Z_{\text{ind}}^1 \neq \emptyset$ . Since by [FSZ, Proposition 1.5 and Theorem 1.6(a)]  $Z_{\text{ind}}^1 = \Phi^{-1}(\mathcal{E})$ , it follows that  $Z_{\text{ind}}^1$  is a  $G_\delta$  in  $Z^1$  and hence a Polish space when endowed with the induced topology. We set

$$Z_{\text{max}}^1 := \{\alpha \in Z^1 \mid \alpha \text{ is quasinormal and } \langle \alpha \rangle = \{\Sigma(J)\}\}.$$

Let  $Q$  be the group of finite permutations of  $J$ ,  $F := \{\tau \in Q \mid \tau(0) = 0\}$ , and  $\mathcal{S}$  the quasinormal subrelation of  $\mathcal{R}$  corresponding to  $F \subset Q$  by Theorem 0.3. Since  $Q$  is dense in  $\Sigma(J)$ , the index cocycle of  $\mathcal{S}$  belongs to  $Z_{\text{max}}^1$  and hence  $Z_{\text{max}}^1 \neq \emptyset$ . Only a slight modification of the routine argument from [PS] or [CHP, Theorem 3] is needed to prove that  $Z_{\text{max}}^1$  is a dense  $G_\delta$  in  $Z^1$ . Since  $Z_{\text{max}}^1 \subset Z_{\text{ind}}^1$ , we obtain

**Proposition 3.1.**  *$Z_{\text{max}}^1$  is a dense  $G_\delta$  in  $Z_{\text{ind}}^1$ .*

In view of this statement it is of interest to give an example of a nonquasinormal ergodic subrelation.

**Example 3.2.** Let  $(X, \mu) = (\{0, 1\}, \lambda)^{\mathbb{Q}}$ , where  $\lambda$  is the equidistribution on  $\{0, 1\}$ . Let  $H := \mathbb{Q} \rtimes \mathbb{Z}$  with multiplication as follows:

$$(q, n)(p, m) = (q + 2^n p, n + m).$$

We define an action of  $H$  on  $X$  by setting  $(hx)_p = x_{2^{-n}(p-q)}$  for all  $p \in \mathbb{Q}$ , where  $h = (q, n)$  and  $x = (x_p)_{p \in \mathbb{Q}}$ . Clearly,  $(X, \mu)$  is an ergodic  $H$ -space. Hence the Cartesian square  $(Z, \nu) = (X, \mu) \times (X, \mu)$  is an ergodic  $H^2$ -space. Denote by  $\mathcal{R}$  the  $H^2$ -orbit equivalence relation. Since  $H$  is amenable,  $\mathcal{R}$  is hyperfinite. Consider the homomorphism  $\pi : H^2 \rightarrow \Sigma(H)$  given by  $\pi(h_1, h_2)(h) = h_1 h h_2^{-1}$ . It is easy to verify that the kernel of  $\pi$  is isomorphic to the center of  $H$ . Since the center is trivial,  $\pi$  is one-to-one. We put

$$W := \{\tau \in \Sigma(H) \mid \tau(1_H) = 1_H, \tau(h') = h', \text{ and } \tau(h'') = h''\},$$

where  $h' = (1, 0)$  and  $h'' = (0, 1)$ . Clearly,  $W$  is an open neighborhood of the identity in  $\Sigma(H)$ . It is routine to verify that  $\pi(s)W \cap W = \emptyset$  for every nontrivial  $s \in H^2$ . Now we define a cocycle  $\sigma : \mathcal{R} \rightarrow \Sigma(H)$  by setting  $\sigma(z, sz) = \pi(s)^{-1}$ ,  $z \in Z$

and  $s \in H^2$ . Let  $\mathcal{S} := \{(z, y) \in \mathcal{R} \mid \sigma(z, y)(1_H) = 1_H\}$ . Notice that  $\mathcal{S}$  is ergodic, since it contains an ergodic subrelation generated by the diagonal (Bernoulli)  $\mathbb{Q}$ -action on  $Z = X \times X$ . For each  $h \in H$ , we define a map  $\phi_h : Z \rightarrow Z$  by setting  $\phi_h(z) = sz$ , where  $s = (1_H, h) \in H^2$ . Then  $\sigma(z, \phi_h(z))(0) = h$ , i.e.  $\{\phi_h\}_{h \in H}$  are choice functions for  $\mathcal{S} \subset \mathcal{R}$  and  $\sigma$  is the corresponding cocycle. We claim that  $\mathcal{S}$  is not quasinormal in  $\mathcal{R}$ . Suppose the contrary: there exists a closed subgroup  $G \subset \Sigma(H)$  and a Borel map  $\phi : Z \rightarrow \Sigma(H)$  such that the cocycle  $\beta : \mathcal{R} \ni (z, y) \mapsto \phi(z)^{-1}\sigma(z, y)\phi(y) \in \Sigma(H)$  takes values and has dense range in  $G$ . Choose an open set  $U \subset \Sigma(H)$  and a neighborhood  $O \subset \Sigma(H)$  of  $\text{Id}_H$  such that  $UOU^{-1} \subset W$  and  $\nu(\phi^{-1}(U)) > 0$ . By assumption, there are a subset  $A \subset Z$  and a nontrivial  $s \in H^2$  with  $\nu(A) > 0$ ,  $A \cap sA \subset \phi^{-1}(U)$ , and  $\beta(sz, z) \in O$  for all  $z \in A$ . Then  $W \not\supset \pi(s) = \sigma(sz, z) \in UOU^{-1} \subset W$  for all  $z \in A$ , a contradiction.

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DEPARTMENT OF MECHANICS AND MATHEMATICS, KHARKOV STATE UNIVERSITY, FREEDOM SQUARE 4, KHARKOV, 310077, UKRAINE

E-mail address: danilenko@ilt.kharkov.ua