ON A CONJECTURE OF F. MÓRICZ

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Abstract. F. Móricz has investigated the integrability of double lacunary sine series. His result, valid for special lacunary sequences, does not extend in the form originally conjectured, but we establish a suitably modified result.

1. Introduction

Let \( a_{ij}, i, j \in \mathbb{N} \), be real numbers satisfying the condition

\[
\sigma = \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^2 \right)^{\frac{1}{2}} < \infty.
\]

Suppose \( q > 1 \) and \( m_i, n_j \) are positive numbers satisfying

\[
\frac{m_{i+1}}{m_i} \geq q, \quad \frac{n_{j+1}}{n_j} \geq q, \quad m_1 = n_1 = 1, \quad i, j \in \mathbb{N}.
\]

Define

\[
f(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \sin m_i x \sin n_j y,
\]

\[
g_j(x) = \sum_{i=1}^{\infty} a_{ij} \sin m_i x, \quad h_j(y) = \sum_{j=1}^{\infty} a_{ij} \sin n_j y.
\]

In the general case these limits are to be understood in the sense of \( L^2 \)-convergence and, as Lemma 1 shows, there is no inherent ambiguity in the definition.

F. Móricz [3] considered the special case when \( m_i = n_i = 2^{i-1}, i \in \mathbb{N} \). In this case he proved that the condition

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \sum_{k=i}^{\infty} \sum_{l=j}^{\infty} a_{kl}^2 \right)^{\frac{1}{2}} < \infty
\]

is equivalent to

\[
\frac{f(x, y)}{xy} \in L(0, 1)^2, \quad \frac{g_i(x)}{x} \in L(0, 1), \quad \frac{h_i(y)}{y} \in L(0, 1), \quad i \in \mathbb{N}.
\]

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He proposed that in the general case when \( m_i, n_j \) are positive integers satisfying condition (2) and the integrability condition, then (4) is satisfied if and only if

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \log \frac{m_{i+1}}{m_i} \log \frac{n_{j+1}}{n_j} \left( \sum_{k=i}^{\infty} \sum_{l=j}^{\infty} a_{kl}^2 \right)^{\frac{1}{2}} < \infty.
\]

Our result is the following

**Theorem.** Let \( a_{ij}, m_i, n_j \) satisfy (1) and (2). Let \( f, g, h \) be as above. Define

\[
S = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|, \quad T = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \log \frac{m_{i+1}}{m_i} \left( \sum_{k=i+1}^{\infty} a_{kj}^2 \right)^{\frac{1}{2}},
\]

\[
U = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \log \frac{n_{j+1}}{n_j} \left( \sum_{l=j+1}^{\infty} a_{il}^2 \right)^{\frac{1}{2}},
\]

\[
V = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \log \frac{m_{i+1}}{m_i} \log \frac{n_{j+1}}{n_j} \left( \sum_{k=i+1}^{\infty} \sum_{l=j+1}^{\infty} a_{kl}^2 \right)^{\frac{1}{2}}.
\]

Then the condition (4) is equivalent to the condition

\[
S + T + U + V < \infty.
\]

We point out that in our theorem, \( m_i, m_j \) need not be integers. If \( m_i = n_i = 2^{i-1} \), then (5) is equivalent to (6). But in general, as the following example shows, (5) is stronger than (6) and they are not equivalent.

First we note that the one-dimensional case is subsumed by the two-dimensional case. If we set \( a_{ij} = b_i \) for \( j = 1 \) and \( a_{ij} = 0 \) for \( j > 1 \), then (5) reads

\[
\sum_{i=1}^{\infty} \log \frac{m_{i+1}}{m_i} \left( \sum_{j=i}^{\infty} b_j^2 \right)^{\frac{1}{2}} < \infty.
\]

and (6) reads

\[
\sum_{i=1}^{\infty} |b_i| + \sum_{i=1}^{\infty} \log \frac{m_{i+1}}{m_i} \left( \sum_{j=i+1}^{\infty} b_j^2 \right)^{\frac{1}{2}} < \infty.
\]

Let \( b_i = \{e^{-2i+1} - e^{-2i+2}\}^{1/2} \) and \( m_i = \prod_{i=1}^{d} e^{2d-1} \). Then we see that (6') holds but (5') does not.

We present the proof of the Theorem in two parts:

**Theorem 1.** Let \( d = 1 + \frac{4}{(q-1)^2} + \frac{2}{(q-1)^2} \sqrt{\frac{4}{(q-1)^2} + 8} \). Then

\[
\int_0^d |g_j(x)| \frac{dx}{x} \leq c_q \left\{ \sum_{i=1}^{\infty} |a_{ij}| + \sum_{i=1}^{\infty} \log \frac{m_{i+1}}{m_i} \left( \sum_{k=i+1}^{\infty} a_{kij}^2 \right)^{\frac{1}{2}} \right\},
\]
the equation:

\[ 1 \]

\[ I \]

\[ d \]

\[ n \]

\[ q \]

\[ S \]

\[ T \]

\[ U \]

\[ V \]

\[ d = d(q) \]

\[ d(q) \]

\[ \int_0^d |h_1(y)| \frac{dy}{y} \leq c_q \left\{ \sum_{j=1}^{\infty} |a_{ij}| + \sum_{j=1}^{\infty} \log \frac{n_{j+1}}{n_j} \left( \sum_{i=j+1}^{\infty} a_{ij}^2 \right)^{\frac{1}{2}} \right\}, \]

\[ \int_0^d \int_0^d |f(x, y)| \frac{dx dy}{xy} \leq c_q (S + T + U + V). \]

This theorem demonstrates that (6) implies (4).

**Theorem 2.** (4) implies (6).

We point out that the number \( d = d(q) \) in Theorem 1 is just the positive root of the equation: \( \frac{1}{4}(d-1)^2 - \frac{2}{(q-1)^2} (d+1) - \frac{4}{(q-1)^2} = 0 \). This particular definition of \( d \) will be retained throughout the paper.

2. Proof of Theorem 1

**Lemma 1.** Let \( a, b \) be arbitrary real numbers and \( Q = (a, a+\alpha) \times (b, b+\alpha) \). Then

\[ 0.001\alpha \leq \left\{ \frac{1}{\sigma^2} \int_Q f^2(x, y) dx dy \right\}^{\frac{1}{2}} \leq \sigma. \]

**Proof.** Let \( I_{ijkl} = \int_Q \sin m_i x \sin m_k x \sin n_j y \sin n_l y \ dx dy \). We have \( \int_Q f^2(x, y) \ dx dy = \sum a_{ijkl} I_{ijkl} \), where the sum \( \sum \) is taken over \( N^4 \). If \( i \neq k \) and \( j \neq l \), then

\[ |I_{ijkl}| \leq \left( \frac{1}{|m_i - m_k|} + \frac{1}{m_i + m_k} \right) \left( \frac{1}{|n_j - n_l|} + \frac{1}{n_j + n_l} \right). \]

Let \( S_i \) denote the subset of \( N^4 \) defined by \( S_i = \{(i, j, k, l) \in N^4 : i \neq k, j \neq l \} \) and let \( \sum_1 \) denote the sum taken over \( S_i \). By Schwarz’s inequality \( |\sum_1 a_{ijkl} I_{ijkl}| \leq \left\{ \sum_1 |a_{ijkl}|^2 \right\}^{\frac{1}{2}} \left\{ \sum_1 I_{ijkl}^2 \right\}^{\frac{1}{2}} \). Applying condition (2) we find \( \sum_1 I_{ijkl}^2 \leq 16(\frac{1}{q-1})^2 \left( \frac{1}{q-1} \right)^2 \) and hence

\[ |\sum_1 a_{ijkl} I_{ijkl}| \leq \left\{ \frac{4}{(q-1)^2} - a_q \right\} \sigma^2, \]

where \( a_q = \frac{4}{(q-1)^2} \).

If \( i = k \) and \( j \neq l \), then \( |I_{ijkl}| \leq \frac{d+1}{2} \left( \frac{1}{|n_j - n_l|} + \frac{1}{n_j + n_l} \right) \). Applying condition (2) again we find \( \sum_2 |a_{ijkl} I_{ijkl}| \leq \left\{ (\frac{2(d+1)}{q-1} - b_q) \right\} \sigma^2 \), where \( \sum_2 \) denotes the sum over the set \( \{i \neq k, j = l\} \cup \{i = k, j \neq l\} \) and \( b_q = \frac{2(d+1)}{(q-1)^2(q+1)}. \)

Finally, if \( (i, j) = (k, l) \), then by condition (2) we have

\[ \frac{1}{4} \left( d - \max \left( \frac{|\sin d|}{q} \right) \right)^2 \leq I_{ijkl} \leq \frac{1}{4} (d+1)^2, \]

\[ \frac{1}{4} (d-1)^2 \sigma^2 \leq \sum_1 \sum_1 \sigma_{ij}^2 I_{ijij} \leq \frac{1}{4} (d+1)^2 \sigma^2. \]

Combining the estimates for \( \sum_1, \sum_2 \) and (10), noticing \( \frac{1}{4} (d-1)^2 - \frac{2(d+1)}{(q-1)^2} - \frac{4}{(q-1)^2} = 0 \) we get \( \sqrt{a_q + b_q} \sigma \leq \{ \int_Q f^2(x, y) \ dx dy \}^{\frac{1}{2}} \leq d \sigma \). If \( 1 < q \leq 31 \), then \( \sqrt{a_q + b_q} \geq \frac{d}{850} \) and hence \( \frac{1}{\sqrt{850}} \int_Q f^2(x, y) \ dx dy \geq \frac{\sigma}{\sqrt{850}} \). If \( q \geq 31 \), then \( d < 1.2 \) and hence \( I_{ijij} \geq \frac{1}{4} (d - \sin 1.2)^2 \). So we modify (10) to get

\[ \int_Q f^2(x, y) \ dx dy \geq \{ a_q + b_q + 0.034 \} \sigma^2 \]
and hence for \( q > 31 \) we have \( \left\{ \frac{1}{\sqrt{q}} \int_Q f^2(x, y) \, dx \, dy \right\}^{\frac{1}{2}} > 0.028 \sigma \). The combination of these estimates completes the proof.

The following lemma is a direct corollary of Lemma 1.

**Lemma 2.** Let \( a_j \) be real numbers satisfying \( A = (\sum_{j=1}^{\infty} a_j^2)^{1/2} < \infty \) and let \( m_j \) be numbers satisfying the condition:

\[ m_1 = 1, \quad \frac{m_{i+1}}{m_i} \geq q > 1. \]

Define \( \psi(x) = \sum_{j=1}^{\infty} a_j \sin m_j x \). Then for any \( a \in \mathbb{R} \)

\[
\frac{1}{200} A \leq \left\{ \frac{1}{d} \int_a^{a+d} \psi^2(x) \, dx \right\}^{\frac{1}{2}} \leq 2A.
\]

**Proof of Theorem 1.** We first prove (7). For fixed \( j \in \mathbb{N} \), we omit the subscript \( j \), and write \( g = g_j \), \( a_i = a_{ij} \) for simplicity. Then we have

\[
\int_0^d x^{-1} |g(x)| \, dx \leq \sum_{i=1}^{\infty} \int_{\frac{d}{m_i}}^{\frac{d}{m_{i+1}}} x^{-1} \left( |S_i(x)| + |T_i(x)| \right) \, dx
\]

where \( S_i(x) = \sum_{k=1}^i a_k \sin m_k x \), \( T_i(x) = \sum_{k=i+1}^{\infty} a_k \sin m_k x \). Since \( |\sin m_k x| \leq m_k x \) we see \( \int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} |S_i(x)| \, dx \leq d \sum_{k=1}^{i} |a_k| \frac{1}{m_i} \), and

\[
\sum_{i=1}^{\infty} \int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} x^{-1} |S_i(x)| \, dx \leq c_q \sum_{i=1}^{\infty} |a_{ij}|.
\]

Simultaneously, writing \( c_k = a_{i+k,j} \), \( u_k = \frac{m_{i+k}}{m_i} \) we get

\[
\int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} |T_i(x)| \, dx = \int_{\frac{d}{m_i}}^{\frac{d}{m_{i+1}}} \left| \sum_{k=1}^{i} c_k \sin u_k x \right| \, dx.
\]

Let \( m = \left\lfloor \frac{m_{i+1}}{m_i} \right\rfloor \) and \( \psi(x) = \sum_{k=1}^{\infty} c_k \sin u_k x \). Then

\[
\int_{\frac{d}{m_i}}^{\frac{d}{m_{i+1}}} |\psi(x)| \, dx \leq \sum_{k=1}^{m} \int_{\frac{d}{m_{i+1}}}^{(\mu+1)d} |\psi(x)| \, dx.
\]

By Lemma 2 we have \( \int_{\mu d}^{(\mu+1)d} |\psi(x)| \, dx \leq 2 \mu (\sum_{k=i+1}^{\infty} a_k^2)^{1/2} \) and hence

\[
\int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} |T_i(x)| \, dx \leq c_q \log \frac{m_{i+1}}{m_i} \left( \sum_{k=i+1}^{\infty} a_k^2 \right)^{1/2},
\]

\[
\sum_{i=1}^{\infty} \int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} x^{-1} |T_i(x)| \, dx \leq c_q \sum_{i=1}^{\infty} \log \frac{m_{i+1}}{m_i} \left( \sum_{k=i+1}^{\infty} a_k^2 \right)^{1/2}.
\]

Combining (12), (13) and (15) we get (7). By symmetry (8) follows.
Now we write $I_{ij} = \int_{m_{i+1}}^{m_i} \int_{n_{j+1}}^{n_j} |f(x, y)| \frac{dxdy}{xy}$ and define
\[
\phi_{kl}(x, y) = a_{kl} \sin m_k x \sin n_l y,
\]
\[
f_1(x, y) = \sum_{k=1}^{i} \sum_{l=1}^{j} \phi_{kl}(x, y), \quad f_2(x, y) = \sum_{k=i+1}^{\infty} \sum_{l=1}^{j} \phi_{kl}(x, y),
\]
\[
f_3(x, y) = \sum_{k=1}^{i} \sum_{l=1}^{\infty} \phi_{kl}(x, y), \quad f_4(x, y) = \sum_{k=i+1}^{\infty} \sum_{l=1}^{\infty} \phi_{kl}(x, y).
\]

Then define
\[
I_{ij}^{(v)} = \int_{d/m_{i+1}}^{d/m_i} \int_{d/n_{j+1}}^{d/n_j} |f_{v}(x, y)| \frac{dxdy}{xy}, \quad v = 1, 2, 3, 4.
\]

We see $I_{ij}^{(1)} \leq d^2 \sum_{k=1}^{i} \sum_{l=1}^{j} |a_{kl}| \frac{1}{q^{2m_i/n}}$, and hence
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} I_{ij}^{(1)} \leq c_q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| = c_q S.
\]

Secondly, $I_{ij}^{(2)} \leq \sum_{i=1}^{j} \frac{d}{q^{2m_i}} \int_{d/m_{i+1}}^{d/m_i} \left| \sum_{k=i+1}^{\infty} a_{kl} \sin m_k x \right| \frac{dx}{x}$. So by (14)
\[
\int_{m_{i+1}}^{m_i} \left| \sum_{k=i+1}^{\infty} a_{kl} \sin m_k x \right| \frac{dx}{x} \leq c_q \log \frac{m_{i+1}}{m_i} \left( \sum_{k=i+1}^{\infty} |a_{kl}|^2 \right)^{1/2}.
\]

Therefore
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} I_{ij}^{(2)} \leq c_q T.
\]

Symmetrically we have
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} I_{ij}^{(3)} \leq c_q U.
\]

Finally, to estimate $I_{ij}^{(4)}$ we change the integrating variables by writing $x = s/m_{i+1}$, $y = t/n_{j+1}$. Then we get
\[
I_{ij}^{(4)} = \int_{d}^{(m_{i+1}/m_i)d} \int_{d}^{(n_{j+1}/n_j)d} \left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl} \sin u_k s \sin v_l t \right| \frac{dsdt}{st},
\]
\[
\text{where } b_{kl} = a_{i+k,j+l}, \quad u_k = \frac{m_{i+k}}{m_i}, \quad v_l = \frac{n_{j+l}}{n_j}.
\]

Let $m = \lfloor m_{i+1}/m_i \rfloor$ and $n = \lfloor n_{j+1}/n_j \rfloor$. Define
\[
\theta_{\mu\nu} = \int_{\mu d}^{(1+\mu)d} \int_{\nu d}^{(1+\nu)d} \left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl} \sin u_k s \sin v_l t \right| \frac{dsdt}{st}.
\]

Applying Lemma 1 we get $\theta_{\mu\nu} \leq \frac{1}{\mu \nu} (\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl}^2)^{1/2}$, and hence
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} I_{ij}^{(4)} \leq c_q V.
\]

A combination of (16)–(19) yields (9). The proof is complete. \[\square\]
3. PROOF OF THEOREM 2

To prove Theorem 2 we need more integral estimates.

**Lemma 3.** Let \( m_i, c_{ij}, i, j \in \mathbb{N} \), be real numbers satisfying condition (11) and \( B = (\sum_{i=1}^{\infty} c_{ii})^2 + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij}^2 < \infty \). Define

\[
h(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} \cos(m_i - m_j)x.
\]

Then, for any \( a \in \mathbb{R} \), \( \int_a^{a+d} h^2(x) \, dx \leq c_q B \), where, and throughout this paper, \( c_q \) denotes a constant depending only on \( q \).

**Proof.** Define

\[
h_1(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} c_{ij} \cos(m_i - m_j)x, \quad h_2(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{j-1} c_{ij} \cos(m_i - m_j)x.
\]

We see \( h^2(x) \leq 3 \{ h_1^2(x) + h_2^2(x) + (\sum_{i=1}^{\infty} c_{ii})^2 \} \). We have \( h_1^2(x) = \frac{1}{2} \{ g_1(x) + g_2(x) \} \) where

\[
g_1(x) = \sum_{i>j} \sum_{k>l} c_{ij} c_{kl} \cos(m_i - m_j - m_k + m_l)x, \quad g_2(x) = \sum_{i>j} \sum_{k>l} c_{ij} c_{kl} \cos(m_i - m_j + m_k - m_l)x.
\]

By condition (11) for \( i > j, k > l \), we have \( m_i - m_j + m_k - m_l \geq 2(q - 1) \cdot \sqrt{m_i - 1 \cdot m_k - 1} \). Hence by Schwarz’s inequality we get

\[
\int_a^{a+d} g_2(x) \, dx \leq c_q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij}^2.
\]

Choose \( n \in \mathbb{N} \) such that \( 1 - \frac{1}{q} - \frac{1}{q^n} = \delta > 0 \). Then define subsets \( J_\mu \) of \( (i, j, k, l) \in \mathbb{N}^4 \) by

\[
J_1 = \{ i > j, i \geq k + n, k > l \}, \quad J_2 = \{ i > j, i \leq k - n, k > l \},
\]

\[
J_3 = \{ k + n > i > k > l, i \geq j + n \}, \quad J_4 = \{ k + n > i > k > l, i > j > i - n \},
\]

\[
J_5 = \{ i + n > k > i > j, k > l \}, \quad J_6 = \{ i = k > j, k > l \}
\]

and define \( g_{1\nu}(x) = \sum_{\nu} c_{ij} c_{kl} \cos(m_i - m_j - m_k + m_l)x, \ \nu = 1, \ldots, 6 \), where the sum \( \sum_{\nu} \) denotes \( \sum_{(i,j,k,l) \in J_\nu} \). Then we see \( g_1 = \sum_{\nu=1}^{6} g_{1\nu} \). First we have \( \int_a^{a+d} g_{11}(x) \, dx \leq c_q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij}^2 \). So by Schwarz’s inequality

\[
\int_a^{a+d} g_{11}(x) \, dx \leq c_q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij}^2.
\]

Symmetrically

\[
\int_a^{a+d} g_{12}(x) \, dx \leq c_q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij}^2.
\]
If \((i, j, k, l) \in J_3\), then \(m_i - m_j - m_k + m_l \geq \delta m_i\). Hence we have
\[
\int_{a}^{a+d} g_{13}(x) \, dx \leq \frac{2}{d} \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} |c_{kl}| \sum_{i=\max(n+1, k+1)}^{\infty} \sum_{j=1}^{i-n} \frac{|c_{ij}|}{m_i}.
\]

The using Schwarz’s inequality we get
\[
\int_{a}^{a+d} g_{13}(x) \, dx \leq c_q B. \tag{23}
\]

For \((i, j, k, l) \in J_4\) let \(\varepsilon_{ijkl} = |f_{a}^{\infty} \cos(m_i - m_j - m_k + m_l) x \, dx|\). Then
\[
\int_{a}^{a+d} g_{14}(x) \, dx \leq \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} \sum_{i=\max(1, k+1)}^{\infty} \sum_{j=1}^{i-n} |c_{ij}| c_{kl} |\varepsilon_{ijkl}|.
\]

If we define \(\varepsilon_{ijkl} = c_{ij} = 0\) when \(i \leq 0\) or \(j \leq 0\), then we get
\[
\int_{a}^{a+d} g_{14}(x) \, dx \leq \sum_{k=2}^{n} \sum_{l=0}^{k-1} \sum_{t=0}^{\infty} |c_{k+l} k+s-t+1 + i| \sum_{t=1}^{k-l} \sum_{l=1}^{c_{k+l} |\eta_t|},
\]

where \(\eta_t(k, s, t) = \varepsilon_{k+s-t+1, k+l} \) will be written as \(\eta_t\) for simplicity. Write \(\sigma_k = \sum_{l=1}^{c} |c_{kl}| \eta_t\). Then \(\sigma_k \leq \left(\sum_{l=1}^{c} c_{kl}^2 \eta_t^2 / n\right)^{1/2}\). Define
\[
l_0 = \min \left\{ l \in \{1, \ldots, k-1\} : |m_{k+s} - m_{k+s-n+1} - m_k + m_l| < \frac{1}{2} \left( 1 - \frac{1}{q} \right) m_l \right\},
\]

\(l_0 = 0\) for the case that the minimum does not exist. As a convention we let \(m_0 = 0\). Then for all \(l \neq l_0\), \(|m_{k+s} - m_{k+s-n+1} - m_k + m_l| \geq \frac{1}{2} (1 - \frac{1}{q}) m_l\). Hence we have \(n_l \leq c_q \frac{1}{m_l}\), for \(l \neq l_0\), \(1 \leq l \leq k-1\), and \(\eta_t \leq n\). Consequently we get \(\sum_{l=1}^{c} \eta_t^2 \sum_{l=1}^{k-l} c_{kl}^2 \eta_t^2 \leq \sum_{l=1}^{c} \eta_t^2 \sum_{l=1}^{k-l} c_{kl}^2 \eta_t^2 \sum_{l=1}^{k-l} c_{kl}^2 \eta_t^2 \sum_{l=1}^{k-l} c_{kl}^2 \eta_t^2 \). Now we derive
\[
\int_{a}^{a+d} g_{14}(x) \, dx \leq c_q B. \tag{24}
\]

Since \(J_5\) is symmetrically related to \(J_3 \cup J_4\) we conclude
\[
\int_{a}^{a+d} g_{15}(x) \, dx \leq c_q B. \tag{25}
\]

By a direct calculation we get
\[
\int_{a}^{a+d} g_{16}(x) \, dx \leq c_q B. \tag{26}
\]

A combination of (21)–(26) yields
\[
\int_{a}^{a+d} g_1(x) \, dx \leq c_q B. \tag{27}
\]

Then we combine (20) and (27) to get \(\int_{a}^{a+d} h_1^2(x) \, dx \leq c_q B\). Symmetrically we conclude \(\int_{a}^{a+d} h_2^2(x) \, dx \leq c_q B\). And at last we derive \(\int_{a}^{a+d} h_2(x) \, dx \leq c_q B\) as required. \(\square\)
Lemma 4. Under the assumptions of Lemma 3 define
\[ g(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} \cos(m_i + m_j)x. \]

Then \( \int_{a}^{a+d} g^2(x) \, dx \leq c_q B. \)

The proof is completely similar to that of Lemma 3. We omit it.

Lemma 5. Under the conditions of Lemma 1
\[ \left\{ \int_Q f^4(x, y) \, dxdy \right\} \leq c_q \sigma. \]

Proof. For \( i \in \mathbb{N} \) and \( y \in \mathbb{R} \) fixed we define \( b_i = \sum_{j=1}^{\infty} a_{ij} \sin n_jy \). Then we see
\[ f^4(x, y) = \frac{1}{2} \{ h(x) + g(x) \} \] where \( h, g \) are defined respectively as in Lemma 3 and Lemma 4 with coefficients \( c_{ij} = b_i b_j \). Then by these lemmas we conclude
\[ \int_Q f^4(x, y) \, dxdy \leq c_q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{b}^{b+d} \{ b_i b_j \}^2 \, dy. \]

Since \( b_i b_j = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{ij} a_{jl} \{ \cos(n_k - n_l)y - \cos(n_k + n_l)y \} \) we apply Lemmas 3 and 4 again to get
\[ \int_{b}^{b+d} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^2 \sum_{i=1}^{\infty} a_{jl}^2 \, dy \leq c_q \sum_{k=1}^{\infty} a_{ik}^2 \sum_{i=1}^{\infty} a_{il}^2. \]

Substituting this into (28) we complete the proof.

The following two estimates follow from Lemma 1 and Lemma 5.

Lemma 6. Under the conditions of Lemma 1 \( c_q \int_Q |f(x, y)| \, dxdy \geq \sigma. \)

Lemma 7. Under the assumptions of Lemma 2 \( c_q \int_{a}^{a+d} |\psi(x)| \, dx \geq A. \)

The following lemma is the essence of Móricz’s Lemmas 2 and 3 of [3].

Lemma 8. Let \( a_i \geq 0, i \in \mathbb{N} \). Then for any \( r \in \mathbb{N} \) and \( n > r \)
\[ \sum_{i=r+1}^{n} a_i \leq \frac{1}{\sqrt{r}} \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2}. \]

Proof of Theorem 2. By Lemma 1, (4) is equivalent to
\[ (4') \quad \frac{f(x, y)}{xy} \in L(0, \lambda d)^2, \quad \frac{g_i(x)}{x} \in L(0, \lambda d), \quad \frac{h_i(y)}{y} \in L(0, \lambda d), \quad i \in \mathbb{N}, \]
where \( \lambda = \max(1, \frac{1}{q-1}) \geq 1 \) and \( d \) is the value defined in Theorem 1. Now assume (4') holds; we are going to prove (6).

Let \( A_{ij} = \log \frac{m_{i+1}}{m_i} \{ \sum_{k=j+1}^{\infty} u_{k j} \}^{1/2}, B_{ij} = \log \frac{m_{j+1}}{m_k} \{ \sum_{l=j+1}^{\infty} u_{ij} \}^{1/2}. \) We first prove \( \sum_{k=1}^{\infty} A_{kj} < \infty, \sum_{l=1}^{\infty} B_{il} < \infty \) \((i, j) \in \mathbb{N} \).

For \( n \) big enough and \( f \) fixed, let \( I_n = \int_{\lambda d/n}^{\lambda d/m_{n+1}} g_j(x) \, dx \). Then \( I_n \geq J_n - R_n \) where
\[ R_n = \sum_{i=1}^{n} \int_{(\lambda d/m_{i+1})}^{(\lambda d/m_i)} \left| \sum_{k=1}^{i} a_{k j} \sin m_kx \right| \frac{dx}{x} \leq c_q \sum_{i=1}^{n} |a_{ij}|. \]
and

\[ J_n = \sum_{i=1}^{n} \int_{\nu d/m_i}^{\infty} \left| \sum_{k=i+1}^{\infty} a_{kj} \sin m_k x \right| \frac{dx}{x}. \]

Applying Lemma 7 we get

\[ J_n \geq \sum_{i=1}^{n} \sum_{\mu=1}^{m_i} \frac{1}{(\mu + \lambda)} c_q \left( \sum_{k=i+1}^{\infty} a_{kj}^2 \right)^{\frac{1}{2}}, \]

where \( m = \lfloor (\frac{m_{i+1}}{m_i} - 1) \lambda \rfloor \in \mathbb{N} \). It is now clear why we take \( \lambda d \) instead of \( d \). We obtain, in fact, \( J_n \geq c_q \sum_{i=1}^{n} A_{ij} \) and hence \( \sum_{i=1}^{n} A_{ij} \leq c_q' \sum_{i=1}^{n} |a_{ij}| + c_q'' I_n \). If \( \sum_{i=1}^{\infty} A_{ij} = \infty \), noticing \( \int_0^{\lambda d} |g_j(x)| \frac{dx}{x} < \infty \) we derive

\[ 1 \leq \lim_{n \to \infty} c_q' \left( \frac{n}{i=1} |a_{ij}| \left( \sum_{i=1}^{n} A_{ij} \right)^{-1} \right). \]

But by Lemma 8 we conclude the right part of this inequality should be zero. This contradiction shows \( \sum_{i=1}^{\infty} A_{ij} < \infty \). Symmetrically we know \( \sum_{j=1}^{\infty} B_{ij} < \infty \).

Next we define \( f_\nu, \nu = 1, 2, 3, 4, \) as in the proof of Theorem 1 and define

\[ E_{ij}^{(\nu)} = \int_{\nu d/m_i+1}^{\lambda d/m_i} \int_{\nu d/n_j+1}^{\lambda d/n_j} |f_\nu(x, y)| \frac{dx dy}{xy}, \quad \nu = 1, 2, 3, 4, \; i, j \in \mathbb{N}. \]

For big \( s, t \in \mathbb{N} \) let

\[ \sigma_{s,t}^{(\nu)} = \sum_{i=1}^{s} \sum_{j=1}^{t} E_{ij}^{(\nu)}, \quad \sigma_{s,t} = \sum_{i=1}^{s} \sum_{j=1}^{t} \int_{\nu d/m_i+1}^{\lambda d/n_j+1} \frac{f(x, y)}{xy} dx dy. \]

We have \( \sigma_{s,t} \geq \sigma_{s,t}^{(4)} - (\sigma_{s,t}^{(1)} + \sigma_{s,t}^{(2)} + \sigma_{s,t}^{(3)}) \). By an argument similar to that used in the proof of Theorem 1 we get inequalities similar to (16)–(18), viz.

\[ \sigma_{s,t}^{(1)} \leq c_g \sum_{i=1}^{s} \sum_{j=1}^{t} |a_{ij}|, \quad \sigma_{s,t}^{(2)} \leq c_g \sum_{i=1}^{s} \sum_{j=1}^{t} \log \frac{m_{i+1}}{m_i} \left( \sum_{k=j+1}^{\infty} a_{kj}^2 \right)^{\frac{1}{2}}, \]

\[ \sigma_{s,t}^{(3)} \leq c_g \sum_{i=1}^{s} \sum_{j=1}^{t} \log \frac{n_{j+1}}{n_j} \left( \sum_{k=j+1}^{\infty} a_{kj}^2 \right)^{\frac{1}{2}}. \]

We now estimate \( \sigma_{s,t}^{(4)} \) applying Lemma 6. Let \( m = \lfloor (\frac{m_{i+1}}{n_j} - 1) \lambda \rfloor \) and \( n = \lfloor (\frac{n_{j+1}}{n_j} - 1) \lambda \rfloor \). Then \( m \in \mathbb{N}, n \in \mathbb{N} \) and

\[ \sigma_{s,t}^{(4)} \geq \sum_{i=1}^{s} \sum_{j=1}^{t} \sum_{\mu=1}^{m_i} \sum_{\nu=1}^{n_j} \int_{\nu d/((\mu - 1)d)}^{\lambda d/((\nu - 1)d)} S_{ij}(x, y) dx dy \]

where

\[ S_{ij}(x, y) = x^{-1} y^{-1} \left| \sum_{k=i+1}^{\infty} \sum_{l=j+1}^{\infty} a_{kl} \sin \frac{m_k}{m_{i+1}} x \sin \frac{n_l}{n_{j+1}} y \right|. \]
Hence by Lemma 6 we get

$$
\sigma_n^{(4)} \geq c_q \sum_{i=1}^{s} \sum_{j=1}^{t} \log \frac{m_i+1}{m_i} \log \frac{n_j+1}{n_j} \left( \sum_{k=i+1}^{\infty} \sum_{l=j+1}^{\infty} a_{kl}^2 \right)^{\frac{1}{2}}.
$$

Then applying Lemma 8, by an argument similar to that for the proof of \( \sum_{i=1}^{\infty} A_{ij} < \infty \) we conclude \( V < \infty \). Then noticing

\[
T = \sum_{i=1}^{\infty} A_{i1} + \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} A_{ij} \leq \sum_{i=1}^{\infty} A_{i1} + c_q V,
\]

\[
U = \sum_{j=1}^{\infty} B_{1j} + \sum_{j=1}^{\infty} \sum_{i=2}^{\infty} B_{ij} \leq \sum_{j=1}^{\infty} B_{1j} + c_q V,
\]

\[
S = \sum_{j=2}^{\infty} |a_{ij}| + \sum_{i=2}^{\infty} |a_{i1}| + |a_{11}| + \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} |a_{ij}|
\]

\[
\leq c_q \sum_{j=1}^{\infty} B_{1j} + c_q \sum_{i=1}^{\infty} A_{i1} + |a_{11}| + c_q V,
\]

we derive (6). The proof is complete. \( \square \)

**Remarks.** (a) In our argument the series by which we define functions need not be trigonometric series because the coefficients \( m_i, n_j \) need not be integers. We must understand such series in the sense of \( L^2 \)-convergence. For example, if conditions (1) and (2) are satisfied, then the “partial sums”

\[
S_{\mu\nu}(x, y) = \sum_{i=1}^{\mu} \sum_{j=1}^{\nu} a_{ij} \sin m_i x \sin n_j y
\]

converge in \( L^2(Q) \) for any compact set \( Q \subset \mathbb{R}^2 \). This is a consequence of Lemma 1. Meanwhile we can easily demonstrate that the convergence of \( S_{\mu\nu} \) in \( L^2(Q) \) does not depend on the manner in which \( \mu \) and \( \nu \) tend to infinity. For a discussion of different kinds of multiple limits we refer the reader to [4].

(b) Since \( S_{\mu\nu} \) can be non-trigonometric sums it does not appear to be a trivial question whether the convergence of \( S_{\mu\nu} \) in \( L^2 \) implies almost everywhere convergence.

(c) Our result can be extended to higher dimensional cases in a quite straightforward manner.

(d) Since this paper was submitted in June, 1993, two related papers of interest have appeared, viz. [1], [2]. We thank the referee for providing the details.

**References**


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