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CENTRALIZERS IN RESIDUALLY FINITE TORSION GROUPS

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In memory of Brian Hartley

Abstract. Let G be a residually finite torsion group. We show that, if G has a finite 2-subgroup whose centralizer is finite, then G is locally finite. We also show that, if G has no 2-torsion, and Q is a finite 2-group acting on G in such a way that the centralizer $C_G(Q)$ is soluble, or of finite exponent, then G is locally finite.

1. Introduction

The study of centralizers in torsion groups and in locally finite groups in particular has been a major research project in the past few decades; see Shunkov [S], Hartley [H1], [H2], [H3, Section 3] and the references therein. The purpose of this note is to apply some powerful Lie-theoretic tools (see Zelmanov [Z2] and Bahturin and Zaicev [BZ]) in the study of centralizers in residually finite torsion groups. Well known constructions by Golod [G], Grigorchuk [Gr], Gupta and Sidki [GS], and others, show that a residually finite torsion group G need not be locally finite. However, the local finiteness of G can be deduced under some extra assumptions on G. For example, if we assume further that G is soluble, or of finite exponent, then G is locally finite; note that the fact residually finite groups of finite exponent are locally finite is one of the equivalent formulations of Zelmanov's celebrated solution to the Restricted Burnside Problem.

As shown by our main result, it sometimes suffices to assume that certain subgroups of G are soluble, or of finite exponent, in order to infer the local finiteness of G. Indeed we have

Theorem 1.1. *Let G be a residually finite torsion group with no 2-torsion, acted on by a finite 2-group Q. Suppose the centralizer $C_G(Q)$ is soluble, or of finite exponent. Then G is locally finite.*

This theorem extends a recent result of Shumyatsky [Sh2], dealing with the case where $|Q| = 2$ and $C_G(Q)$ is abelian; it also extends Zelmanov's solution to the Restricted Burnside Problem for groups of odd exponent (take $Q = 1$ in Theorem 1.1).

The assumption that G has no 2-torsion is not required if the centralizer $C_G(Q)$ is finite. Indeed, using Theorem 1.1 we can easily deduce the following.

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Corollary 1.2. *Let G be a residually finite torsion group, and suppose G has a finite 2-subgroup Q whose centralizer $C_G(Q)$ is finite. Then G is locally finite.*

Note that, if Q has order 2, then the conclusion already follows from Shunkov's Theorem [S] (see also Hartley's excellent survey [H1]). The cases $Q = C_2 \times C_2$ and $Q = C_{2^n}$ have recently been settled in [Sh1] and [RSh].

For residually nilpotent groups we obtain a bit more. By a recent result of Zelmanov, residually nilpotent torsion groups satisfying an identity are locally finite (this follows from Theorem 1.6 of [Z2]). In the following result we draw a similar conclusion assuming only that the centralizer of a certain automorphism group satisfies an identity.

Proposition 1.3. *Let G be a residually nilpotent torsion group with no 2-torsion, acted on by a finite 2-group Q. Suppose the centralizer $C_G(Q)$ satisfies some non-trivial identity. Then G is locally finite.*

Our proof of these results is surprisingly short, and is essentially Lie-theoretic. Lie methods are often used in studying fixed points of group automorphisms (see for instance [HB, Chapter 8]); however, this note seems to be the first application of Zelmanov's work to this subject.

I am grateful to Y. Bahturin and to P. Shumyatsky for sending me their preprints [BZ] and [RSh]; results from these papers play an important role in this note.

2. Proofs

In what follows we assume that G is a torsion group with no 2-torsion, and that Q is a finite 2-group acting on G. Let $C = C_G(Q)$ be the centralizer of Q in G. We shall show that, under some suitable assumptions, G is locally finite.

Our starting point is the following lemma by Rocco and Shumyatsky [RSh, Lemma 2.2].

Lemma 2.1. *Let G, Q be as above, and let $N \lhd G$ be a Q-invariant normal subgroup. Then $C_{G/N}(Q) = C_G(Q)N/N$.*

We note that the assumption that Q is a 2-group is essential to the proof of the lemma.

Now suppose G is residually p for some prime $p \neq 2$, and that the centralizer $C_G(Q)$ satisfies some non-trivial group identity. We assume that G is finitely generated, and aim to show it is finite.

Let $D_n = D_n(G)$ be the nth dimension subgroup of G in characteristic p. The series $\{D_n\}_{n\geq 1}$, also known as the Zassenhaus-Jennings-Lazard series (see [HB, Chapter 8]), is a central series with elementary abelian factors. Consider the associated Lie algebra

$$L(G) = \bigoplus_{n\geq 1} D_n/D_{n+1},$$

defined over the field $\mathbb{F}_p$ with p elements. Then Q acts on $L(G)$ by Lie algebra automorphisms. Moreover, Lemma 2.1 shows that the centralizer $C_{L(G)}(Q)$ of Q in the Lie algebra is the subalgebra of $L(G)$ corresponding to C, namely

$$C_{L(G)}(Q) = \bigoplus_{n\geq 1} (C \cap D_n)D_{n+1}/D_{n+1}.$$

Since $C_G(Q)$ satisfies some group identity, the Lie subalgebra $C_{L(G)}(Q)$ is PI (i.e., satisfies some non-trivial Lie identity).

We record the following recent result of Bahturin and Zaicev [BZ, Theorem 2].

Theorem 2.2. *Let L be a Lie algebra over a field F, acted on by a finite soluble group H whose order is not divisible by the characteristic of F. Suppose the centralizer $C_L(H)$ is PI. Then L is PI.*

Applying this, we conclude that our Lie algebra $L(G)$ is PI. Next, we use the following result of Zelmanov [Z2, Theorem 1.6].

Theorem 2.3. *Let G be a finitely generated periodic residually p group. Suppose the Lie algebra $L(G)$ constructed as above is PI. Then G is finite.*

We note that the proof of this result combines a hard theorem of Zelmanov on the nilpotency of some PI Lie algebras [Z2, Theorem 1.7] (see also [Z1, Proposition 2] for an important special case) with Lazard's work on p-adic analytic groups [L]. The idea is to show first that $L(G)$ is nilpotent, and then to deduce that the pro-p completion of G is p-adic analytic, hence linear. This shows that G itself is linear. The conclusion of Theorem 2.3 then follows from the local finiteness of torsion linear groups.

Applying Theorem 2.3 we see that our group G is finite. This proves Proposition 1.3 for residually p groups. Now, if G is residually nilpotent with no 2-torsion, then it decomposes into a direct product of maximal p-subgroups ($p \neq 2$), which are Q-invariant and residually p. It follows that the maximal p-subgroups of G are locally finite, so G is locally finite.

Proposition 1.3 is proved.

In order to prove Theorem 1.1 we need to quote a result of Thompson [T]. We denote the Fitting height (i.e., the nilpotent length) of a soluble group H by $h(H)$. For a positive integer n, let $l(n)$ denote the number of primes (counted with multiplicity) appearing in the prime factorization of n.

Theorem 2.4. *Let H be a finite soluble group acted on by a finite group K, and suppose $(|H|, |K|) = 1$. Let $l(|K|) = k$. Then $h(H) \leq 5^k h(C_H(K))$.*

See also [K] and [Tu] for improvements of this bound.

We can now prove Theorem 1.1. Suppose first that the centralizer $C = C_G(Q)$ is soluble, and denote its Fitting height by l. Let $N \lhd G$ be a Q-invariant subgroup of finite index, and consider the factor group G/N, which is acted on by Q. Then G/N has odd order, hence it is soluble [FT]. By Lemma 2.1 we have $C_{G/N}(Q) = CN/N$. It follows that $h(C_{G/N}(Q)) \leq h(C) = l$. Suppose Q has order 2^k and set $m = 5^k l$. Applying Thompson's Theorem we obtain

$$h(G/N) \leq m.$$

We conclude that G is residually (finite of Fitting height $\leq m$). It follows that G has a series of characteristic subgroups

$$G = G_0 \geq G_1 \geq \ldots \geq G_m = 1,$$

such that the factors G_i/G_{i+1} are all residually nilpotent.

Clearly, Q acts on each factor G_i/G_{i+1} in such a way that the centralizer $C_{G_i/G_{i+1}}(Q)$ is soluble (being isomorphic to a section of C). Applying Proposition 1.3 we conclude that the groups G_i/G_{i+1} are all locally finite. Hence G is locally finite.

Note that if we replace the solubility of C by the assumption that C satisfies an identity w with the property that finite soluble groups satisfying w have bounded

Fitting height, then all our arguments go through. By the following consequence of the Hall-Higman theory, this is the case if w is the identity $x^n = 1$.

Lemma 2.5. *The Fitting height of finite soluble groups of exponent n is bounded above by some function of n.*

Proof. Write $n = p_1^{e_1} \cdots p_k^{e_k}$ where p_i are distinct primes, and let H be a finite soluble group of exponent n. Then the p_i-length $l_{p_i}(H)$ of H is at most $2e_i$ (see [HB, Chapter 9]), and this yields

$$h(H) < \prod_{i=1}^{k}(l_{p_i}(H) + 1) \leq \prod_{i=1}^{k}(2e_i + 1).$$

The result follows. □

The proof of Theorem 1.1 is now complete.

Remark. In a similar manner, it suffices to assume in Theorem 1.1 that $C_G(Q)$ satisfies an Engel condition, or that $C_G(Q)$ has a finite normal series such that each of its factors is either soluble or of finite exponent, in order to deduce the local finiteness of G.

More generally, it would be interesting to find out which identities bound the Fitting height of finite soluble groups, and thereby to produce various variations on Theorem 1.1.

Finally, let us prove Corollary 1.2. Let G be a residually finite torsion group, and let $Q \leq G$ be a finite 2-subgroup whose centralizer $C_G(Q)$ is finite. Then G has a finite index Q-invariant subgroup G_1 such that $G_1 \cap C_G(Q) = 1$. Hence Q acts on G_1 and $C_{G_1}(Q) = 1$. By [RSh, Lemma 2.1], G_1 has no 2-torsion. Applying Theorem 1.1 (using the solubility of the trivial group!) we conclude that G_1 is locally finite. Therefore G is locally finite, as required.

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