PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 12, December 1998, Pages 3633–3637
S 0002-9939(98)04517-1

BANACH SPACES FAILING THE ALMOST ISOMETRIC UNIVERSAL EXTENSION PROPERTY

D. M. SPEEGLE

(Communicated by Dale Alspach)

Abstract. If X is an infinite dimensional, separable, uniformly smooth Banach space, then there is an $\epsilon > 0$, a Banach space Y containing X as a closed subspace and a norm one map T from X to a $C(K)$ space which does not extend to an operator $\tilde{T}$ from Y to $C(K)$ with $\|\tilde{T}\| \leq 1 + \epsilon$.

A pair of Banach spaces (E, X) with E a closed subspace of X is said to have the *λ-into-$C(K)$ extension property* (λ-EP for short) if for every $C(K)$ space, and every linear map $T : E \to C(K)$, there is an extension $\tilde{T} : X \to C(K)$ of T such that $\|\tilde{T}\| \leq \lambda\|T\|$. We will say that a separable space E has the *λ-universal extension property* (λ-UEP) if (E, X) has the λ-EP whenever E imbeds as a (closed) subspace of a separable space X. We restrict to separable X to rule out known counterexamples, namely $X = \ell_\infty$.

The best known result along these lines is Sobczyk's theorem [S], which in this language says that c_0 has the 2-UEP. In addition, Lindenstrauss and Pelczynski [LP] proved that whenever E is a closed subspace of c_0, (E, c_0) has the $(1 + \epsilon)$-EP for all $\epsilon > 0$, a fact which was generalized to $(E, c_0(\Gamma))$ by Johnson and Zippin [JZ]. Since c_0 is separably injective, it follows that any subspace E of c_0 has the $(2 + \epsilon)$-UEP for all $\epsilon > 0$. (It is not hard to show that, in fact, if (E, X) has the M-EP and X has the N-UEP, then E has the NM-UEP; see below.)

Turning now to spaces which have the 1-UEP, we see that the situation is completely solved. A result of Kalman's implies (cf. [K],[L]) that if the dual ball of a finite dimensional Banach space E has only finitely many extreme points, then E has the 1-UEP. In [L], Lindenstrauss proved the converse; namely, if a Banach space has the 1-UEP, then the dual ball of E can have only finitely many extreme points.

However, the almost isometric case is still open. It is easy to see from Kalman's result that all finite dimensional spaces have the $(1 + \epsilon)$-UEP for all $\epsilon > 0$, which is the only positive result in this direction to date.

In this paper we investigate whether finite dimensional spaces are the only spaces which have the $(1+\epsilon)-$UEP for all $\epsilon > 0$, which we refer to as the *almost isometric universal extension property* (AIUEP). Using a technique similar to one in [L], we

Received by the editors December 23, 1996 and, in revised form, April 25, 1997.
1991 *Mathematics Subject Classification.* Primary 46B20.
The author was supported in part by the NSF through the Workshop in Linear Analysis and Probability at Texas A&M.

are able to imbed some Banach spaces into the continuous functions on a weak*-closed subset of its dual ball in such a way that this pair fails the $(1+\epsilon)$-EP. In particular, we get that c_0 and all infinite dimensional, separable, uniformly smooth spaces fail the AIUEP.

We begin by recalling that there is a correspondence between norm-one maps from a Banach space E to a $C(K)$ space and continuous functions from K to the unit ball (B_{E^*}, w^*). The application of this fact in the context of the extension property yields the following proposition, which can be found in [Z].

Proposition 1 ([Z]). *Let E be a closed subspace of the Banach space X, and let $\lambda \geq 1$. Then (E, X) has the λ-EP if and only if there is a weak*-weak* continuous function $f : B_{E^*} \to \lambda B_{X^*}$ which extends functionals, i.e. $f(e^*)|_E = e^*$.*

A minor modificaton of Proposition 1 gives a simple necessary condition for the pair (E, X) to have the λ-EP.

Proposition 2. *Let E be a closed subspace of the Banach space X, and let $\lambda \in \mathbb{R}$. If (E, X) has the λ-EP, then every weak* null sequence $\{e_n^*\}$ in B_{E^*} has a weak* null sequence $\{x_n^*\}$ of extensions to X^* satisfying $\|x_n^*\| \leq \lambda$.*

Proof. By Proposition 1, $E \subset X$ has the λ-EP if and only if there is a weak*-weak* continuous map $f : B_{E^*} \to \lambda B_{X^*}$ which extends functionals.

Now, notice that if f is such a function, so is $g(e^*) = \frac{f(e^*) - f(-e^*)}{2}$. In particular, there is such a function satisfying $g(e^*) = -g(-e^*)$. Now, suppose $E \subset X$ has the λ-EP, and let $e_n^* \to 0$. Then, $-e_n^* \to 0$ as well. So, $g(-e_n^*) \to g(0)$, and $g(-e_n^*) = -g(e_n^*) \to -g(0)$. Hence, $g(0) = 0$, and in order for g to be continuous, $\{g(e_n^*)\}_{n=1}^\infty$ is weak* null. □

We are now ready to give a condition which implies that a Banach space E fails the AIUEP. Recall that a *weak* slice* of the unit ball B_{E^*} is a set $S(e, \alpha) = \{e^* \in B_{E^*} : e^*(e) > \alpha\}$. We say a Banach space has *property $\mathcal{A}$ with constant α* if there is a normalized weak* null sequence $\{e_n^*\}$, an $\alpha \in [0, 1)$, and a normalized sequence $\{e_n\} \in E$ such that (1) $e_n^*(e_n) \to 1$ and (2) $S(e_n, \alpha) \cap -S(e_m, \alpha) = \emptyset$ for all n, m.

Theorem 3. *Let E be a Banach space which satisfies property $\mathcal{A}$ with constant α, and let $\epsilon < \frac{1-\alpha}{1+\alpha}$. Then, E does not have the $(1+\epsilon)$ universal extension property. Moreover, there is a space X containing E as a codimension one subspace such that (E, X) fails the $(1+\epsilon)$-EP.*

Proof. Let $\{e_n^*\}$ and $\{e_n\}$ be sequences satisfying the conditions of property $\mathcal{A}$. Let $K = B_{E^*} \setminus (\bigcup_{n=1}^\infty S(e_n, \alpha))$. We will show that $(E, C(K))$ does not have the $(1+\epsilon)$ extension property. Consider the map $E \to C(K, w^*)$ under the natural map $e \to \hat{e}$, where $\hat{e}(e^*) = e^*(e)$.

The first step is to verify that the natural map is an isometric imbedding. Since K is weak* closed, it suffices to show that the weak* closed symmetric convex hull of K is all of B_{E^*}. To do this, it suffices then to show that the symmetric hull of K contains S_{E^*}. So, suppose e^* is a norm one functional in E^* which is not in K. Then, e^* is in S_n for some n. So, by property $\mathcal{A}$, $-e^*$ is not in S_m for any m and $K \cup -K$ contains S_{E^*}.

To show that E fails the almost isometric universal extension property, we will show that E together with the above imbedding into $C(K)$ fails the conditions of Proposition 2. To this end, let μ_n be extensions of e_n^* to functionals on $C(K)$ with norm less than $1+\epsilon$, and let $f : K \to \mathbb{R}$ be the constant one function. We show that $\mu_n(f)$ is bounded away from 0.

Write μ_n as $\mu_n^+ - \mu_n^-$ via the Hahn decomposition [R]. Then, for some $\beta_n \geq 0$,

$$
\begin{aligned}
1-\beta_n &= \int_K \hat{e}_n d\mu_n \\
&\leq \int_{K_1^n \cup K_2^n} \alpha d\mu_n^+ + \int_{K_1^n \cup K_2^n} d\mu_n^- \\
&= \alpha\mu_n^+(K_1^n) + \mu_n^-(K_2^n), \qquad (1)
\end{aligned}
$$

where K_1^n and K_2^n correspond to the positive and negative parts of μ_n on K.

We also have, by the norm of μ_n, that

$$\mu_n^+(K_1^n) + \mu_n^-(K_2^n) \leq 1+\epsilon. \qquad (2)$$

Adding $\frac{-2}{1-\alpha}$ times (1) to $\frac{1+\alpha}{1-\alpha}$ times (2), we can compute

$$
\begin{aligned}
\mu_n(f) &= \int_K f d\mu_n \\
&= \mu_n^+(K_1^n) - \mu_n^-(K_2^n) \\
&= \frac{-2}{1-\alpha}(\alpha\mu_n^+(K_1^n) + \mu_n^-(K_2^n)) + \frac{1+\alpha}{1-\alpha}(\mu_n^+(K_1^n) + \mu_n^-(K_2^n)) \\
&\leq \frac{-2}{1-\alpha}(1-\beta_n) + \frac{1+\alpha}{1-\alpha}(1+\epsilon) \\
&= -1 + \frac{1+\alpha}{1-\alpha}\epsilon + \frac{2\beta_n}{1-\alpha}.
\end{aligned}
$$

So, since $\beta_n \to 0$ and $\epsilon < \frac{1-\alpha}{1+\alpha}$, we have that for large n, $\mu_n(f) \leq c < 0$ for some constant c, as desired.

To prove the last statement, instead of looking at $E \to C(K)$, we could just as well have considered E imbedded into $\overline{sp}(E \cup \{f\})$ in $C(K)$, and the proof goes through. Thus, E imbeds as a codimension one subspace of a space X such that (E, X) fails the almost isometric universal extension property. □

By examining the proof of Theorem 3 more carefully, we gain the following additional information. Since weak* null sequences in B_{E^*} correspond to maps from E to c_0, it follows that spaces with property $\mathcal{A}$ actually fail the $(1+\epsilon)$-into-c_0 extension property. Of course, by Sobczyk's theorem [S], every norm-one map from a subspace of a separable space into c_0 extends to a map of norm less than or equal to 2.

Proposition 4. *If X is an infinite dimensional, separable, uniformly smooth Banach space, then X has property $\mathcal{A}$ with constant α for some $\alpha \in [0, 1)$.*

Proof. We construct the necessary sequences $\{y_n^*\}$, $\{y_n\}$, and the constant α. Let $\{x_n\}$ be a sequence which is dense in B_X. Let y_1^* be in the unit sphere S_{X^*} of B_{X^*}, and let y_1 be a norm one element in X such that $y_1^*(y_1) = 1$. Let $S_1 = \{x^* \in S_{X^*} : |x^*(y_1)| < \frac{1}{2}, x^*(x_1) < \frac{1}{2}\}$. Then, $S_1 \neq \emptyset$, so let $y_2^* \in S_1$, and let $y_2 \in S_X$ norm y_2^*. Continuing in this fashion, we get $y_i^* \in S_i = \{x^* \in S_{X^*} : |x^*(y_j)| < \frac{1}{i+1}, x^*(x_j) < \frac{1}{i+1}$ for all $1 \leq j < i\}$, where $y_j \in S_X$ satisfy $y_j^*(y_j) = 1$. To define the constant α, let δ denote the modulus of convexity of X^*. We claim that $\alpha = 1 - \delta(\frac{1}{4})$

together with the above $\{y_i\}$ and $\{y_i^*\}$ satisfies the properties required for X to have property $\mathcal{A}$. (Note that since X^* is uniformly convex, $\alpha \in [0,1)$.)

To prove the claim, first note that since $\{x_n\}$ is dense in B_X, the sequence $\{y_n^*\}$ converges to 0 in the weak* topology. So, it remains to prove that $S(y_i,\alpha) \cap -S(y_j,\alpha) = \emptyset$. For $i = j$, this is immediate, while for $j < i$, we have $\|y_i + y_j\| \geq |y_i^*(y_i + y_j)| \geq 1 - \frac{1}{i+1} \geq \frac{1}{2}$. We also have that the diameter of the slice $S(y_i,\alpha)$ is less than $\frac{1}{4}$. Hence, if $x \in S(y_i,\alpha) \cap -S(y_j,\alpha)$, it follows that $\|y_i - x\| < \frac{1}{4}$ and $\|x + y_j\| < \frac{1}{4}$; hence, $\|y_i + y_j\| < \frac{1}{2}$; therefore, $S(y_i,\alpha) \cap -S(y_j,\alpha) = \emptyset$. □

We note here that the result of Theorem 3 is the best possible when $E = c_0$. Indeed, if we consider e_n to be the standard unit vector basis for c_0, it is easy to see that, if we consider slices of the extreme points of B_{ℓ_1} rather than slices of B_{ℓ_1}, then c_0 has property $\mathcal{A}$ with constant α for all $\alpha > 0$; hence, c_0 fails the $(2-\epsilon)$-UEP for all $\epsilon > 0$. (It is easy to see that, in fact, (c_0, c) fails the $(2-\epsilon)$-EP for all $\epsilon > 0$.) However, we do not know in general if finding the best possible α yields the best possible λ for which (E, X) has the λ-into-c_0 extension property, even when E is assumed to be a codimension one subspace of X.

Finally, we prove the fact mentioned in the introduction.

Proposition 5. *Let X be a separable Banach space with the N-UEP, and let (E, X) have the M-EP. Then, E has the MN-UEP.*

Proof. Let $T : E \to C(K)$ be a bounded linear operator, let $i : E \to X$ denote the imbedding of E into X, and let $j : E \to Y$ denote an imbedding into an arbitrary separable Banach space Y. The proof follows from taking pushouts (cf. [J] for definitions and properties) as in the following commutative diagram. All of the maps along the outer commuting square are isometric imbeddings, and P is separable.

$$\begin{tikzcd}
E \arrow[rr, "i"] \arrow[dr, "T"] \arrow[dd, "j"'] & & X \arrow[dl, "\tilde{T}"'] \arrow[dd, "k"] \\
& C(K) & \\
Y \arrow[rr, "l"] \arrow[ur, "\hat{T}|Y"] & & P \arrow[ul, "\hat{T}"']
\end{tikzcd}$$

The map $\tilde{T}$ is an extension of T to X such that $\|\tilde{T}\| \leq M\|T\|$, and $\hat{T}$ is an extension of $\tilde{T}$ to P such that $\|\hat{T}\| \leq N\|\tilde{T}\| \leq MN\|T\|$. The restriction of $\hat{T}$ to Y is the desired extension of T from E to Y. □

Acknowledgment

The author would like to thank W. B. Johnson for suggesting improvements to the author's original argument. In particular, Proposition 2 was suggested by him.

References

[J] W. B. Johnson, *Extensions of c_0*, Positivity (to appear).

[JZ] W. B. Johnson and M. Zippin, *Extensions of operators from subspaces of $c_0(\gamma)$ into $C(K)$ spaces*, Proc. AMS **107** (1989), 751-754. MR **90b:**46045

[K] J. A. Kalman, *Continuity and convexity of projections and barycentric coordinates in convex polyhedra*, Pacific J. Math **22** (1961), 1017-1022. MR **24:**A3557

[L] J. Lindenstrauss, *Extension of compact operators*, Memoirs of the AMS **48** (1964). MR **31:**3828

[LP] J. Lindenstrauss and A. Pelczynski, *Contributions to the theory of classical Banach spaces*, J. Functional Analysis **8** (1971), 225-249. MR **45:**863

[R] H. L. Royden, *Real Analysis 3rd Ed.*, Macmillan, New York, 1988. MR **90g:**00004

[S] A. Sobczyk, *Projection of the space m on its subspace c_0*, Bull. Amer. Math. Soc. **47** (1941), 938-947. MR **3:**205f

[Z] M. Zippin, *A global approach to certain operator extension problems*, Lecture Notes in Math. **1470** (1991), 78-94. MR **93b:**47011

Department of Mathematics, Texas A & M University, College Station, Texas 77843

E-mail address: speegle@math.tamu.edu

Current address: Department of Mathematics, Saint Louis University, Saint Louis, Missouri 63103