A MAXIMUM PRINCIPLE FOR P-HARMONIC MAPS
WITH L^q FINITE ENERGY

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Abstract. We show a maximum principle for P-harmonic maps with L^q-
finite energy. As an application we can generalize a non-existence theorem
for harmonic maps with finite Dirichlet integral by Schoen and Yau to those
maps.

1. Introduction

Let f : (M, g) → (N, h) be a smooth map of Riemannian manifolds. We are
interested in determining under which condition the boundedness of the length
|df| of df can be induced. This problem is related to the maximum principle for
solutions of an elliptic operator of second order in a sense. For instance several
Schwarz type lemmas for harmonic maps, which state the boundedness of
|df| under a certain negative curvature condition of the target manifold (N, h), are known and
a few Liouville type theorems are induced from them (cf. [C-Y], [E-L], [G-H], [Sh],
[Y-2]). As a degenerate case of the negativity condition of the curvature of (N, h)
the following observation is interesting in view of our aspect (see also [C]).

Theorem*. Let f : (M, g) → (N, h) be a smooth map from a non-compact com-
plete (connected) Riemannian manifold (M, g) of dimension m into a Riemannian
manifold (N, h). Let \( \Gamma_M \) (resp. \( \Gamma_N \)) denote the set of points of M (resp. N) where
Ric_M (resp. Riem_N) is not non-negative (resp. not non-positive). Let \( \Sigma \) and E
be subsets of M such that \( \Sigma := \Gamma_M \cup f^{-1}(\Gamma_N) \cup E \) and \( M \setminus \Sigma \neq \emptyset \). Suppose
f : (M \setminus \Sigma, g) → (N, h) is harmonic, i.e. Trace_g(\nabla (df)) = 0 on M \setminus \Sigma, and \( \int_{M \setminus \Sigma} |df|^q < +\infty \) for \( q > 1 \). Then the following holds:

\[
|df|(x) \leq \sup_{y \in \Sigma} |df|(y) \quad \text{for any } x \in M,
\]

where the right hand side of the above inequality may take infinity and is defined to
be zero if \( \Sigma = \emptyset \).

To show this result the subharmonicity of \( |df| \) on \( M \setminus \Sigma \) plays an essential role
in view of the Weitzenböck formula. In fact Theorem* can be obtained by [Sc-Y],
Theorem 1, and applying the following general statement to \( |df| \) (see also [L-S]).

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Proposition. Let \((M, g)\) be as above and let \(v\) be a smooth function on \(M\) satisfying 
\[
\Delta v := \text{Trace}_g \nabla^2 v \geq 0 \quad \text{outside a proper subset } A \text{ of } M.
\]
Suppose the interior of \(\{v \leq \sup_A v\}\) is not empty and \(\int_{M \setminus A} |dv^q| < +\infty\) with \(q \geq 1\). Then \(v(x) \leq \sup_A v\) for any \(x \in M\).

For the proof the reader should see [Y-1], Corollary and the proof of Theorem 3 (see also §2). Here the \(L^1\)-integrability condition of \(dv^q\) is essential and cannot be removed.

For a positive number \(p > 1\) \(f : (M, g) \to (N, h)\) is said to be \(p\)-harmonic if \(f\) satisfies the following:

\[
\text{Trace}_g \nabla (|df|^p - 2df) = 0 \quad \text{on } M.
\]

As in the case of a harmonic map, i.e. \(p = 2\), any \(p\)-harmonic map \(f : (M, g) \to (N, h)\) can be characterized as a critical point of the \(L^p\) energy functional \(\int_M |df|^p\) if it is finite. In this note we generalize Theorem* to \(p\)-harmonic maps. However we cannot apply the above proposition to the case \(p \neq 2\) because the subharmonicity of \(|df|^2\) breaks down. Nevertheless we will show the following by modifying an idea developed in [T].

Theorem. Let \(f : (M, g) \to (N, h)\) and \(\Sigma\) be as in Theorem*. Suppose \(f : (M \setminus \Sigma, g) \to (N, h)\) is \(p\)-harmonic, i.e. \(\text{Trace}_g \nabla (|df|^q - 2df) = 0\) on \(M \setminus \Sigma\), and \(\int_{M \setminus \Sigma} |df|^q < +\infty\) for \(q > p - 1\) with \(p \geq 2\). Then the following holds:

\[
|df|(x) \leq \sup_{y \in \Sigma} |df|(y) \quad \text{for any } x \in M.
\]

This result can be regarded as a maximum principle for \(L^q\) integrable solutions of a non-linear degenerate elliptic equation. In particular, we obtain the following (cf. [L-S], [N], [Sc-Y]).

Corollary. Let \((M, g)\) and \((N, h)\) be a non-compact complete (connected) Riemannian manifold with non-negative Ricci curvature and a Riemannian manifold with non-positive Riemannian curvature respectively. Then any \(p\)-harmonic map \(f : (M, g) \to (N, h)\) with \(L^q\) finite energy is constant if \(q > p - 1\) with \(p \geq 2\) \((q \geq 1\) when \(p = 2\)).

2. Proof of the Theorem

By the Weitzenböck formula (cf. [E-L], (2.20) and (3.13)) we obtain the following:

\[
\frac{1}{2} \Delta |df|^2 = |\nabla (df)|^2 - \langle df, (DD^* + D^*D)(df) \rangle + Q(df),
\]

where \(D^*\) is the adjoint operator of

\[
D : \Gamma(\bigwedge^s TM^* \otimes f^*TN) \to \Gamma(\bigwedge^{s+1} TM^* \otimes f^*TN)
\]

and \(Q(df)\) is given by the following:

\[
Q(df) := \sum_{i=1}^m \langle \text{Ric}_M(df(e_i)), df(e_i) \rangle - \sum_{j,k=1}^m \langle \text{Riem}_N(df(e_j), df(e_k))df(e_k), df(e_j) \rangle.
\]
Since $Q(df) \geq 0$ on $M \setminus \Sigma$ by the curvature condition and $D(df) = 0$, we obtain for any real number $q$

\[
\frac{1}{2}|df|^q \Delta |df|^2 + |df|^q \langle df, DD^*(df) \rangle \\
\geq |df|^q |\nabla (df)|^2 \quad \text{on} \quad M \setminus (\Sigma \cup \{ |df| = 0 \}).
\]

(1)

If $c_0 := \sup_{y \in \Sigma} |df|(y) = +\infty$, then the assertion is trivial. We have only to consider the case $0 \leq c_0 < +\infty$. Let $M = M_* \cup M_+$, $M_* := \{ |df| \leq c_0 \}$, and $M_+ := \{ |df| > c_0 \}$. Assume $M_+ \neq \emptyset$. First we take a smooth non-negative function $\lambda$ on a real line satisfying $\lambda \equiv 0$ on $(-\infty, c_0^2]$, $\lambda > 0, \lambda' \geq 0$ on $(c_0^2, +\infty)$ and $\lambda \equiv 1$ on $[c_1, +\infty)$, $c_1 > c_0^2$. By the completeness of $g$, for a fixed point $x_* \in M$ and any $r > 0$ there exists a non-negative Lefschetz continuous function $\omega$ on $M$ such that $0 \leq \omega \leq 1$ on $M$, $\omega \equiv 1$ on $B_r(x_*)$, $\omega \equiv 0$ on $M \setminus B_{2r}(x_*)$, and $|d\omega| \leq C/r$ on $M$ for $C > 0$ not depending on $r$, where $B_r(x_*)$ is the geodesic ball centered at $x_* \in M$ of radius $r > 0$. Setting $u := |df|$ it should be noted that $\Sigma \cup \{ u = 0 \} \subset M_*$, $u$ is smooth on $M_+$, and $\lambda(u(x)^2) > 0$ if and only if $x \in M_+$. By (1) we have

\[
\int \lambda(u^2)\omega^2 u^{q-2} |\nabla (df)|^2 \leq \frac{1}{2} \int \lambda(u^2)\omega^2 u^{q-2} \Delta u^2 \\
+ \int \lambda(u^2)\omega^2 u^{q-2} \langle df, DD^*(df) \rangle.
\]

(2)

Using integration by parts we have

\[
\int \lambda(u^2)\omega^2 u^{q-2} \Delta u^2 = - \int \langle \nabla(\lambda(u^2)\omega^2 u^{q-2}), \nabla u^2 \rangle \\
= - \int \lambda'(u^2)\omega^2 u^{q-2} |\nabla u^2|^2 - 2 \int \lambda(u^2)\omega u^{q-2} \langle \nabla \omega, \nabla u^2 \rangle \\
- \int \lambda(u^2)\omega^2 \langle \nabla u^{q-2}, \nabla u^2 \rangle \\
\leq 2 \int \lambda(u^2)\omega^2 u^{q-2} |\nabla \omega|^2 - 2(q - 2 - \varepsilon) \int \lambda(u^2)\omega^2 u^{q-2} |\nabla u|^2
\]

(3)

for any $q > 0$ and $\varepsilon > 0$. Here we remark the following:

\[
D^*(vu^{p-2}df) = vD^*(u^{p-2}df) - u^{p-2}e(dv)^* df
\]

on $\{ u > 0 \}$ for any real number $p$ and any smooth function $v$ on $M$, where $e(dv)^*$ is the adjoint operator of the left exterior multiplication by $dv$. In particular, by the $p$-harmonicity of $f$, i.e. $D^*(u^{p-2}df) = 0$, on $M_+$ we obtain

\[
D^*(vu^{q-2}df) = -u^{p-2}e(d(vu^{q-p}))^* df,
\]

(3)

\[
D^*(df) = (p-2)e(d\log u)^* df
\]

(4)
on $M_+$ for any $p > 1$ and $q > 0$. Using integration by parts, and the Cauchy-Schwarz inequalities $|\langle du\rangle| \leq |\nabla u|$ and $|D^* (df)| \leq \sqrt{m} |\nabla (df)|$, we obtain
\[
\int \lambda (u^2 ) \omega^2 u^{q - 2} \langle df, DD^* (df) \rangle = \int (D^* (\lambda (u^2 ) \omega^2 u^{q - 2} df), D^* (df)) \leq - \int \{ 2 \lambda' (u^2 ) u^{q - 1} + (\lambda (u^2 ) u^{q - 3} \lambda (u^2 ) \} \langle du \rangle |D^* (df)| + 2 \int \lambda (u^2 ) \omega u^{q - 2} \langle du \rangle |D^* (df)| \quad \text{by (3)}
\]
\[
(*) \quad \leq - \min \{ 0, (q - p)(p - 2) \} \int \lambda (u^2 ) u^{q - 2} \omega^2 |\nabla u|^2 + 2 \int \lambda (u^2 ) \omega u^{q - 2} \langle du \rangle |D^* (df)| \quad \text{by (4)}
\]
\[
\leq - \min \{ 0, (q - p)(p - 2) \} \int \lambda (u^2 ) u^{q - 2} \omega^2 |\nabla u|^2 + \epsilon \int \lambda (u^2 ) \omega u^{q - 2} |\nabla (df)|^2 + \frac{m}{\epsilon} \int \lambda (u^2 ) u^q |\nabla \omega|^2
\]
for any $q > 0$ and $\epsilon > 0$. Since $|\nabla u| \leq |\nabla (df)|$ on $\{ u > 0 \}$, by (2), (*), and (**), we obtain for any $q > 0$ and $0 < \epsilon \leq 1$
\[
(q - 1 + \min \{ 0, (q - p)(p - 2) \} - 2\epsilon) \int \lambda (u^2 ) \omega^2 u^{q - 2} |\nabla u|^2 
\]
\[
\leq \frac{m}{\epsilon} \int \lambda (u^2 ) u^q |\nabla \omega|^2.
\]
If $q > p - 1$ with $p > 2$, then $\epsilon := \min \{ 1, q - 1 + \min \{ 0, (q - p)(p - 2) \} \}/3 > 0$.
Since $\lambda' \geq 0$ and $\sup \lambda' = 1$, we obtain
\[
\int_{B_r (x_0)} \lambda (u^2 ) u^{q - 2} |\nabla u|^2 \leq \frac{(m + 1) C^2}{\epsilon^2 r^2} \int_{M \setminus \Sigma} u^q.
\]
By the $L^q$ integrability condition of $u$ on $M \setminus \Sigma$ and letting $r$ tend to infinity we obtain
\[
\int_M \lambda (u^2 ) u^{q - 2} |\nabla u|^2 = 0.
\]
Since $\lambda \geq 0$ and $\lambda (u(x)^2 ) > 0$ if and only if $x \in M_+$, the above equality implies $\nabla u \equiv 0$ on $M_+$, i.e. $u = |df| \equiv \text{constant} > c_0$ on $M_+$ by the connectedness of $M$. If $\Sigma \neq \emptyset$, then by continuity there exists a point $x_0 \in M_+$ such that $|df| (x_0 ) > c_0$, which is a contradiction and $M_+$ is empty. If $\Sigma = \emptyset$, i.e. $c_0 = 0$, then $u \equiv c > 0$ on $M$. Hence by [Y-1], Theorem 7, $\int_M |df|^q = c^q \times \text{Vol} (M) = +\infty$, which is also a contradiction and so $M_+$ is empty, i.e. $f$ is constant. This completes the proof of Theorem. In case of $p = 2$ and $q = 1$, the assertion of Corollary follows from [L-S], Theorem 2.2, (b).

References


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