STABILITY OF THE WULFF SHAPE

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Abstract. We consider the functional of a hypersurface, given by a convex elliptic integrand with a volume constraint. We show that, up to homothety and translation, the only closed, oriented, stable critical point is the Wulff shape.

1. Introduction

Let $F : S^n \rightarrow \mathbb{R}^+$ be a smooth function and denote its gradient with respect to the standard metric on $S^n$ by $DF$. Consider the map

$$\phi : S^n \rightarrow \mathbb{R}^{n+1},$$

$$\nu \rightarrow F\nu + DF,$$

and consider the (possibly singular) hypersurface defined by $W_F := \phi(S^n)$. The differential of $\phi$ is given by

$$d\phi_\nu = (D^2F + F1)_\nu,$$

where $D^2F$ denotes the intrinsic Hessian of $F$ on $S^n$ and 1 denotes the identity on $T_\nu S^n$. If we impose the condition that

$$D^2F + F1 > 0, \quad \forall \nu \in S^n,$$

where $> 0$ means that the matrix is positive definite, then $W_F$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of $F$. We assume from now on that this is the case.

Now let

$$X : \Sigma \rightarrow \mathbb{R}^{n+1}$$

be a smooth immersion of a compact, oriented hypersurface, possibly with nonempty boundary. Let

$$\nu : \Sigma \rightarrow S^n$$

denote its Gauss map. We assign to each such $X$ the functional

$$J_F(X) := \int_\Sigma F(\nu) dA_X.$$
Such functionals have been treated by many authors (see for example [1], page 517, and the references therein). The algebraic \((n+1)\)-volume enclosed by \(\Sigma\) is given by
\[
V(X) = \frac{1}{n+1} \int_\Sigma \langle X, \nu \rangle dA_X.
\]
We are interested in those hypersurfaces which are critical points of the functional \(J_F\) restricted to those hypersurfaces enclosing a fixed volume \(V\) (and spanning a fixed boundary in the case that \(\partial \Sigma \neq \emptyset\)). By a standard argument involving Lagrange multipliers, this means we are considering critical points of the functional
\[
J_{F, \Lambda}(X) := \int_\Sigma F(\nu) dA_X + \Lambda V(X),
\]
where \(\Lambda\) is a constant. We will show below that such critica are characterized by the Euler-Lagrange equation
\[
div_\Sigma DF - nh F + \Lambda = 0
\]
where \(h\) is the mean curvature of \(\Sigma\). In (5) the constant \(\Lambda = \Lambda_\Sigma\) can be determined for a critical immersion.

We will call a critical immersion \(X\) stable if and only if the second variation of \(J_F\) (or equivalently of \(J_{F, \Lambda}\)) is non-negative for all (compactly supported) variations of \(X\) preserving the enclosed \((n+1)\)-volume. We have the following result.

**Theorem 1.** Let \(X : \Sigma \rightarrow \mathbb{R}^{n+1}\) be a smooth immersion of an oriented, closed \((\partial \Sigma = \emptyset)\), stable critical point of \(J_{F, \Lambda}\). Then, up to translation, \(X(\Sigma) = tW_F\), where \(t = -n/\Lambda_\Sigma\).

A result known as Wulff’s Theorem states that \(W_F\) actually minimizes \(J_F\) among all hypersurfaces having the same volume, so the stability of \(W_F\) is obvious. Proofs of Wulff’s Theorem can be found in [3] and [4].

Note that the case \(F \equiv 1\) is the widely studied class of constant mean curvature hypersurfaces. For this special case Theorem 1 was first obtained by Barbosa and do Carmo [2]. Later, Wente [5] gave a closely related but more direct proof which avoided the use of the Jacobi operator. The proof of Theorem 1 is a modification of Wente’s idea to the case of non-constant \(F\).

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2. Preliminaries

Let \(X : \Sigma \rightarrow \mathbb{R}^{n+1}\) be a smooth oriented hypersurface with Gauss map \(\nu : \Sigma \rightarrow S^n\). Consider a variation of \(X\) of the form
\[
X_t = X + t(\psi \nu + \xi) + O(t^2)
\]
where \(\psi\) is a smooth function on \(\Sigma\) and \(\xi\) is tangent to \(X\). (Both \(\psi\) and \(\xi\) are assumed to have compact support if \(\partial \Sigma \neq \emptyset\).) The corresponding first variation of the normal is given by
\[
\delta \nu = -\nabla \psi + d\nu(\xi)
\]
and the first variation of the volume element is
\[
\delta dA = (\text{div} \xi - nh \psi) dA.
\]
We use this to compute the first variation of $J_F$,
\[
\delta J_F(X) = \int_{\Sigma} (\langle DF(\nu), \delta \nu \rangle + F(\nu)\delta dA) \\
= \int_{\Sigma} (\langle DF(\nu), -\nabla \psi \rangle - nhF\psi dA) \\
+ \int_{\Sigma} (\langle DF(\nu), d\nu(\xi) \rangle + F\text{div}\xi)dA.
\]
Note that $T_{\nu(p)}S^n = dX(T_p \Sigma)$, so that if we denote the inner product on this plane by $\langle \cdot, \cdot \rangle$, then
\[(6) \quad \langle (\nabla F \circ \nu), \cdot \rangle = d(F \circ \nu) = \nu^* dF = \langle DF(\nu), d\nu(\cdot) \rangle = \langle d\nu(DF), \cdot \rangle.
\]
From this it follows that
\[(7) \quad \nabla(F \circ \nu) = d\nu(DF)
\]
Integrating by parts, we then obtain
\[
\delta J_F(X) = \int_{\Sigma} (\psi(\text{div}DF - nhF))dA + \int_{\Sigma} (\langle d\nu(DF), \xi \rangle + F\text{div}\xi)dA \\
= \int_{\Sigma} (\psi(\text{div}DF - nhF))dA + \int_{\Sigma} (\langle \nabla F, \xi \rangle + F\text{div}\xi)dA \\
= \int_{\Sigma} (\psi(\text{div}DF - nhF))dA + \int_{\Sigma} (-F\text{div}\xi + F\text{div}\xi)dA \\
= \int_{\Sigma} (\psi(\text{div}DF - nhF))dA.
\]
Finally, we recall that the first variation of the volume $V(X)$ is
\[
\delta V(X) = \int_{\Sigma} \psi dA
\]
Hence we obtain the Euler-Lagrange equation for the functional $J_F, \Lambda$:
\[(8) \quad \text{div}DF - nhF + \Lambda = 0.
\]
Under the convexity condition (2) this equation is elliptic. In fact its linearization is given by the elliptic, self-adjoint Jacobi operator
\[L[\psi] = \text{div}((D^2 F + F1)\nabla \psi) + \psi(d\nu, (D^2 F + F1) \circ d\nu).
\]
Later we will need the following lemma

**Lemma 2.** Let $X : \Sigma \rightarrow \mathbb{R}^{n+1}$ be an oriented, smooth, critical immersion of $J_{F, \Lambda}$. Then
\[(9) \quad (n + 1)\Lambda V(X) = -n \int_{\Sigma} F dA.
\]
**Proof.** Compute
\[(10) \quad \text{div}(F\nabla|X|^2/2) = \langle \nabla F, \nabla|X|^2/2 \rangle + F\Delta|X|^2/2.
\]
A well-known formula gives
\[(11) \quad \Delta|X|^2/2 = nh\sigma + n,
\]
where $\sigma = \langle X, \nu \rangle$ is the support function. Therefore
\[
\text{div}(F\nabla|X|^2/2) = \langle \nabla F, \nabla|X|^2/2 \rangle + nh\sigma F + nF
\]
\[
= \langle d\nu(DF), \nabla|X|^2/2 \rangle + nh\sigma F + nF
\]
\[
= \langle DF, d\nu(\nabla|X|^2/2) \rangle + nh\sigma F + nF
\]
\[
= \langle DF, \nabla \sigma \rangle + nh\sigma F + nF,
\]
using the fact that $d\nu(\nabla|X|^2/2) = \nabla \sigma$. On the other hand,
\[
\text{div}(\sigma DF) = \langle \nabla \sigma, DF \rangle + \sigma \text{div}DF = \langle \nabla \sigma, DF \rangle + nh\sigma F - \Lambda \sigma.
\]
It then follows that
\[
0 = \int_\Sigma \text{div}(F(\nabla|X|^2/2) - \sigma DF)dA = \int_\Sigma (nF + \Lambda \sigma)dA = n\int_\Sigma FdA + (n + 1)V(X).
\]

3. Main result

Let $X : \Sigma \to \mathbb{R}^{n+1}$ be as above. We first consider the variation of $X$ given by
\[
X_t = X + t\phi(\nu) = X + t(F\nu + DF).
\]
We then determine a function $s(t)$ such that
\[
\tilde{X}_t := s(t)X_t
\]
satisfies
\[
\tilde{X}_0 = X, \quad V(\tilde{X}_t) \equiv \text{const}.
\]
We then show that $\partial^2_{tt}J_F(\tilde{X}_t) \leq 0$ holds with equality if and only if the image of $X$ is a multiple of $W_F$.

Clearly,
\[
J_F(\tilde{X}_t) = J_F(sX_t) = s^n J(X_t),
\]
so that
\[
\partial^2_{tt}J_F(\tilde{X}_t)_{t=0} = (ns''(0) + n(n - 1)(s'(0))^2)J_F(X)
\]
\[
+ 2ns'(0)\partial_t J_F(X)_{t=0} + \partial^2_{tt}J_F(X)_{t=0}.
\]
Note that
\[
dX_t = dX + td(\phi \circ \nu) = dX + t((D^2F + F1) \circ d\nu) \perp \nu.
\]
This implies that if $\nu_t$ is the Gauss map of $X_t$ then $\nu_t \equiv \nu$. It follows that
\[
\partial_t J_F(X)_{t=0} = \int_\Sigma F(\nu)\partial_t (dA_{X_t})_{t=0}
\]
and
\[
\partial^2_{tt}J_F(X)_{t=0} = \int_\Sigma F(\nu)\partial_{tt}(dA_{X_t})_{t=0}.
\]
Observe that since $\partial_t(X_t)_{t=0} = DF + F\nu$, we have
\[
\partial_t (dA_{X_t})_{t=0} = (\text{div}DF - nhF)dA = -\Lambda dA.
\]
In the following lemma we will compute the coefficients in (13).

**Lemma 3.** $s'(0) = \Lambda/n$, and $s''(0) = 2\Lambda^2/n^2$. 

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Proof. Recall that \( s(t) \) is defined by the condition
\[
V(\tilde{X}_t) = s^{n+1}(t) V(X_t) \equiv V(X).
\]
Differentiating this gives
\[
(n+1)s'(t)s^n(t) V(X_t) + s^{n+1}(t) \partial_t V(X_t) \equiv 0.
\]
At any time \( t \), the first variation of the volume is
\[
\partial_t V(X_t) = \int_\Sigma \langle \partial_t X, \nu \rangle dA_{X_t} = \int_\Sigma F(\nu) dA_{X_t},
\]
since the normal is constant in \( t \). In particular, when \( t = 0 \) this gives
\[
\partial_t V(X_t)_{t=0} = \int_\Sigma F(\nu) dA_X,
\]
so \( s'(0) = \Lambda/n \) by Lemma 2. Also, differentiating (19) and setting \( t = 0 \) gives
\[
((n+1)s''(0) + n(n+1)(s'(0))^2)V(X)
+ 2(n+1)s'(0)\partial_t V(X_t)_{t=0} + \partial^2_{tt} V(X_t)_{t=0} = 0.
\]
It then follows from Lemma 2, (20), and (17) that \( s''(0) = 2\Lambda^2/n^2 \). \( \Box \)

Using the lemma and (13), we obtain
\[
\partial^2_{tt} J_F(\tilde{X}_t)_{t=0} = \int_\Sigma (F(A^2(\frac{1}{n}-1)dA + \partial^2_{tt}(dA_{X_t})_{t=0})).
\]
Let \( \{e_i\}_{i=1,...,n} \) be a local orthonormal frame for the metric induced by \( X \) defined on a neighborhood in \( \Sigma \). Let \( g_t = (g_{i,j}(t)) \) be the matrix representation of the metric induced by \( X_t \) with respect to this frame, so that locally
\[
dA_{X_t} = (detg_t)^{1/2}dA_X.
\]
Since \( g_0 = id \) we have
\[
(detg_t)'(0) = tr(g'(0)) = -2\Lambda,
\]
(24)
\[
(detg_t)''(0) = (tr(g'(0)))^2 + tr(g''(0)) - tr(g'(0))2,
\]
so that
\[
((detg_t)^{1/2})''(0) = -\frac{1}{4}(trg')^2 + \frac{1}{2}(trg')^2 + \frac{1}{2}(trg'') - \frac{1}{2}tr(g')^2
= \Lambda^2 + \frac{1}{2}(trg') - tr(g')^2.
\]
Locally, we have
\[
g'_{i,j} = \langle X'_i, X_j \rangle + \langle X_i, X'_j \rangle,
\]
\[
g''_{i,j} = 2\langle X'_i, X'_j \rangle.
\]
Let \( A_{i,j} = \langle X'_i, X_j \rangle \). Then by using the fact that \( dX' \) is tangential, we have
\[
tr(g')^2 = tr(A + A')^2 = |A + A'|^2 \quad tr(g'') = 2|A|^2.
\]
Therefore
\[
\frac{1}{2}(trg'') - \frac{1}{2}tr(g')^2 = \frac{1}{2}(2 \sum_{i,j} A^2_{i,j} - \sum_{i,j} (A_{i,j} + A_{j,i})^2
= - \sum_{i,j} (A_{i,j} A_{j,i}).
\]
Therefore in (23) we have, locally,

\[ F(\Lambda^2 \left( \frac{1}{n} - 1 \right) dA + \partial^2_{\nu}(dA_X))_{t=0} = \Lambda^2 / n - \sum_{i,j} (A_{i,j} A_{j,i}). \]

(26)

We now choose our frame more carefully. Fix \( p \in \Sigma \) and note that \( dX(T_p \Sigma) = T_{\nu(p)} S^n \). We choose an orthonormal frame so that \( \{ e_i(p) \} \) diagonalizes \( D^2 F(\nu(p)) \). This is possible since the Hessian is symmetric. With respect to this basis, the matrix representing \( (D^2 F + F1) \nu \) has the form \( \text{diag}(\mu_1, ..., \mu_n) \), with \( \mu_i > 0 \) for all \( i \) by the convexity condition. Let \( (\nu_{i,j}) \) be the (symmetric) matrix representing \( d\nu(p) \) with respect to the given basis. Then at \( p \) we have

\[ \langle X'_i, X'_j \rangle = A_{i,j} = (\text{diag}(\mu_1, ..., \mu_n)(d\nu))_{i,j} = (\mu_i \nu_{i,j}). \]

Then

\[
\Lambda^2 / n - \sum_{i,j} (A_{i,j} A_{j,i}) = 1/n(\sum_{i=1,...,n} \mu_i \nu_{i,i})^2 - \sum_{i,j=1,...,n} \mu_i \mu_j \nu_{i,j}^2 \\
\leq (1/n)n \sum_{i=1,...,n} \mu_i^2 \nu_{i,i}^2 - \sum_{i=1,...,n} \mu_i^2 \nu_{i,i}^2 - \sum_{i \neq j} \mu_i \mu_j \nu_{i,j}^2 \\
= -\sum_{i \neq j} \mu_i \mu_j \nu_{i,j}^2 \leq 0.
\]

Equality holds between the first two lines if and only if \( \mu_i \nu_{i,i} \equiv t, i = 1, ..., n, \) for some number \( t \). Equality holds in the last line if and only if \( \nu_{i,j} = 0 \) for all \( i \neq j \). If this is the case, then

\[ \sum \mu_i \nu_{i,i} = nt = -\Lambda \Sigma, \]

so \( t \) does not depend on the point. Therefore the integrand in (26) is non-positive and vanishes identically if and only if at each point

(27)

\[ (D^2 F + F1) \nu \equiv -(\Lambda \Sigma / n)1. \]

However, this implies that \( \Sigma \) is convex and in particular that \( \nu \) is a diffeomorphism of \( \Sigma \) onto \( S^n \). Let \( \sigma \) denote the support function of \( \Sigma \) and let \( \tilde{\sigma} := \sigma \circ \nu^{-1} \). Then the map \( \nu^{-1} \) is a reparametrization of \( X(\Sigma) \), and it can be expressed as

(28)

\[ \nu^{-1} := (D\tilde{\sigma} + \tilde{\sigma} \nu) : S^n \rightarrow \mathbb{R}^n. \]

Therefore we have

\[ 1 = d\nu^{-1} \circ d\nu = (D^2 \tilde{\sigma} + \tilde{\sigma} 1) \circ d\nu. \]

It then follows from (27) that

\[ D^2(\tilde{\sigma} + F/\Lambda) + (\tilde{\sigma} + F/\Lambda 1) \equiv 0, \]

so that integrating gives

\[ D(\tilde{\sigma} + F/\Lambda) + (\tilde{\sigma} + F/\Lambda) \nu \equiv \bar{c} = \text{const}. \]

The conclusion then follows.
REFERENCES


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