REAL FORMS OF A RIEMANN SURFACE OF EVEN GENUS

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ABSTRACT. Natanzon proved that a Riemann surface $X$ of genus $g \geq 2$ has at most $2(\sqrt{g} + 1)$ conjugacy classes of symmetries, and this bound is attained for infinitely many genera $g$. The aim of this note is to prove that a Riemann surface of even genus $g$ has at most four conjugacy classes of symmetries and this bound is attained for an arbitrary even $g$ as well. An equivalent formulation in terms of algebraic curves is that a complex curve of an even genus $g$ has at most four real forms which are not birationally equivalent.

1. INTRODUCTION

Natanzon [4] (see also [3]) proved that a Riemann surface $X$ of genus $g \geq 2$ has at most $2(\sqrt{g} + 1)$ conjugacy classes of symmetries, and this bound is attained for infinitely many odd genera $g$. Singerman [5] showed that if $X$ is hyperelliptic, then the number of non-conjugate pairs of symmetries does not exceed 3 if $g$ is even and 4 if $g$ is odd. The aim of this note is to prove that a Riemann surface of even genus $g$ has at most 4 conjugacy classes of symmetries and this bound is attained for an arbitrary even $g$ as well. An equivalent formulation is that a Riemann surface of even genus $g$ is the complex double of at most 4 bordered Klein surfaces, or in terms of algebraic curves, that a complex curve of an even genus $g$ has at most four real forms which are not birationally equivalent (see [1]).

2. PRELIMINARIES

Let $X$ be a Riemann surface of genus $g \geq 2$. By a symmetry of $X$ we mean an anticonformal involution $\sigma$ of $X$ with fixed points and a surface admitting a symmetry is said to be symmetric. A symmetric Riemann surface $X$ corresponds to a real algebraic curve. In the group $\text{Aut}^\pm(X)$, of all conformal and anticonformal automorphisms of $X$, non-conjugate symmetries correspond to different real models of the curve.

Arbitrary compact Riemann surfaces of genus $g \geq 2$ can be represented as the orbit space $\mathcal{H}/\Gamma$ of the hyperbolic plane $\mathcal{H}$ with respect to the action of a Fuchsian surface group $\Gamma$, a discrete subgroup of $\text{Aut}^\pm(\mathcal{H}) = \text{PSL}(2, \mathbb{R})$ without elliptic elements. A discrete subgroup $\Lambda$ of $\text{Aut}^\pm(\mathcal{H})$ with compact orbit space is called an
NEC (non-euclidean crystallographic) group. The algebraic structure of an NEC group Λ is determined by the signature:

\[ s(Λ) = (h; ±; [m_1, ..., m_r]; \{ (n_{11}, ..., n_{is_1}), ..., (n_{ik}, ..., n_{ks_k}) \}). \]

The orbit space \( H/Λ \) is an orbifold with underlying surface of genus \( h \), having \( r \) cone points and \( k \) boundary components, each with \( s_j \geq 0 \) corner points. The signs “+” and “−” correspond to orientable and non-orientable orbifolds respectively. The integers \( m_i \) are called the proper periods of \( Λ \) and they are the orders of the cone points of \( H/Λ \). The integers \( n_{ij} \) are the link periods of \( Λ \); they are the orders of the boundary components, each with \( s_i \) corner points. The brackets \( (\cdot) \) of \( Λ \) are determined by the signature:

\[ \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{s_i} \left( 1 - \frac{1}{n_{ij}} \right). \]

where \( ε = 2 \) if there is a “+” sign and \( ε = 1 \) otherwise. If \( Λ' \) is a subgroup of \( Λ \) of finite index, then it is an NEC group itself and the following Riemann-Hurwitz formula holds:

\[ [Λ : Λ'] = \frac{μ(Λ')}{μ(Λ)}. \]

Given an NEC group \( Λ \) the subgroup \( Λ^+ \) of \( Λ \) consisting of the orientation-preserving elements is called the cannonical Fuchsian subgroup of \( Λ \).

An NEC group \( Γ \) without elliptic elements is called a surface group and it has signature \( (h; ±; [-], \{(-), k, (-)\}) \). In such a case \( H/Γ \) is a Klein surface, i.e., a surface with a dianalytic structure of topological genus \( h \), orientable or not according to whether the sign is “+” or “−” and having \( k \) boundary components. Conversely, a Klein surface whose complex double has genus greater than one can be expressed as \( H/Γ \) for some NEC surface group \( Γ \). Furthermore, given a Riemann (resp. Klein) surface represented as the orbit space \( X = H/Γ \), with \( Γ \) a surface group, a finite group \( G \) is a group of automorphisms of \( X \) if and only if \( G = Λ/Γ \) for some NEC group \( Λ \).

3. 2-GROUPS OF AUTOMORPHISMS OF SURFACES OF EVEN GENUS

We start this section with the following lemma which allows us to restrict ourselves to finite 2-groups when we work with conjugacy classes of symmetries of a Riemann surface.

Lemma 3.1. Let \( g \) be a positive integer greater than or equal to 2. There exists a Riemann surface \( X \) of genus \( g \) having \( k \) non-conjugate symmetries which generate a 2-subgroup of \( \text{Aut}^+(X) \) such that any Riemann surface \( X' \) of genus \( g \) has no more than \( k \) conjugacy classes of symmetries.
Proof. Let $X$ be a Riemann surface of genus $g$ with maximal number $k$ of non-conjugate symmetries $\tau_1, \ldots, \tau_k$ and let $G_2$ be a 2-Sylow subgroup of $G = \text{Aut}^+(X)$. Then by the Sylow theorem $\tau_1^{a_1}, \ldots, \tau_k^{a_m} \in G_2$, for some $a_1, \ldots, a_m \in G$, and hence the result.

Lemma 3.2. Let $X$ be a Riemann surface of even genus $g$ and let $G$ be a 2-group of automorphisms of $X$. Then $G$ is an extension of a cyclic or dihedral group by $Z_2$. In particular, if $G$ is generated by symmetries, then either $G$ is a cyclic group, a dihedral group, or a semidirect product of a cyclic or dihedral group by $Z_2$.

Proof. Let $X = \mathcal{H}/\Gamma$ for some surface NEC group $\Gamma$ and let $G = \Lambda/\Gamma$. Assume that $\Lambda$ has a signature of a general form (1). By Theorems 2.2.4 and 2.3.3 of [2], $G$ is cyclic or dihedral if $\Lambda$ has a proper period is equal to $|G|$ or a link period equal to $|G|/2$ respectively. So assume that neither a proper period of $\Lambda$ is equal to $|G|$ nor a link period is equal to $|G|/2$. Let $M = \{(i, j) \mid 2m_i = |G|\}$ and $N = \{(i, j) \mid 4n_{ij} = |G|\}$, and let $N'$ and $M'$ be the complementary sets. Then

$$\frac{|G|}{2} \left(\alpha h - 2 + k + \sum_{i \in M'} \left(1 - \frac{1}{m_i}\right) + \sum_{(i, j) \in N'} \frac{1}{2} \left(1 - \frac{1}{n_{ij}}\right)\right)$$

is even and, since $g$ is even, we obtain from the Riemann-Hurwitz formula (3) that

$$\frac{|G|}{2} \left(\sum_{i \in M} \left(1 - \frac{1}{m_i}\right) + \sum_{(i, j) \in N} \frac{1}{2} \left(1 - \frac{1}{n_{ij}}\right)\right)$$

is odd. But the last is equal to $|G||M|/2 + |G||N|/4 - |M| - |N|$ and therefore $|M| + |N|$ is odd. So in particular $|M| \neq 0$ or $|N| \neq 0$. In the first case $G$ contains $H = Z_{|G|/2}$ while in the second one $H = D_{|G|/4}$ as a subgroup of index 2, hence the first part of the theorem. Now, if in addition $G$ is generated by elements of order 2, then there exists an element $g \in G \setminus H$ of order 2 and so $G = H \ltimes Z_2$. This completes the proof.

Theorem 3.3 (Main result). A Riemann surface of even genus $g$ has at most 4 conjugacy classes of symmetries. Furthermore this bound is attained for every even genus $g$.

Proof. The dihedral 2-group has three conjugacy classes of elements of order 2 and it is easy to check that all semidirect products $G = Z_n \ltimes Z_2$ have at most three such classes. So it remains to count the number of conjugacy classes of symmetries of Riemann surfaces whose groups of automorphisms are semidirect products $G = D_n \ltimes Z_2$, where $n$ is a power of 2. These groups have the presentations:

$$G_{\alpha, \beta} = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^n = 1, zxz = (xy)^\alpha x, zyz = (xy)^\beta x \rangle,$$

where $\alpha - \beta \equiv 1 \mod 2$, $\alpha(\alpha - \beta + 1) \equiv 0 \mod n$, $\beta(\alpha - \beta) + \alpha + 1 \equiv 0 \mod n$.

From the proof of the previous lemma it follows that we can assume that $x, y$ and $z$ are symmetries of $X$. So neither $(xy)^{n/2}$ nor an element of the form $z(xy)^{\delta}x$ can be a symmetry since they preserve the orientation of $X$ as compositions of an even number of symmetries.

Finally an element of the form $z(xy)^\gamma$ has order two if and only if

$$\gamma(\alpha - \beta + 1) \equiv 0 \mod n.$$
We have 
\[(xy)^m z(xy)^{-m} = z(xy)^{m(\alpha - \beta - 1)} \text{ and } x(xy)^m z(xy)^{-m} x = z(xy)^{m(\beta + 1 - \alpha) + \alpha}\]
and so \(z(xy)^{2s\alpha}\) and \(z(xy)^{(2s+1)\alpha}\) are conjugate to \(z\) for all \(s\). In particular for \(\alpha\) odd there is only one conjugacy class of elements of order 2 of the form \(z(xy)\).

Now let \(\alpha\) be even and let \(\alpha - \beta + 1 = 2^at\), where \(t\) is odd.

If \(s = 0\), then \(\alpha - \beta + 1 = 0\) and so arbitrary \(z(xy)^\gamma\) has order 2. Furthermore, \((xy)^{-m} z(xy)^m = z(xy)^{2m}, (xy)^{-m} z(xy)(xy)^m = z(xy)^{2m+1}\) and so there are at most two conjugacy classes of elements of order 2 of this form with representatives \(z\) and \(zxy\). But actually these elements are non-conjugate in \(G\) since also \(x(xy)^m z(xy)^{m(\alpha - \beta - 1)} = z(xy)^{2m+\alpha}\), where \(2m+\alpha\) is even. If \(s = 1\), then \(\alpha = n/2\) or \(\alpha = 0\) and \(z, z(xy)^{n/2}\) are the only elements of order 2 of this form and furthermore they are non-conjugate just for \(\alpha = 0\) and \(\beta = n - 1\), i.e., for \(G = D_n \times Z_2\). Indeed for \(\alpha \neq 0\), \(xxy = z(xy)^\alpha\) and, for \(\alpha = 0, \beta \neq n - 1, yzy = z(xy)^\beta + 1\). Finally, if \(s \geq 2\), then every element of order 2 of this form is equal to \(z(xy)^{2v}\) for some \(v\). On the other hand \(\alpha - \beta - 1 = 2u\), for some odd \(u\), and thus every element of the form \(z(xy)^{2v}\) is conjugate to \(z\). This completes the proof of the first part.

To prove the second part of the theorem, observe that by [6] the signature of the canonical Fuchsian subgroup of an NEC-group with signature \((0;+,\{+1,2\})\) is maximal. So by Remark 5.1.1(1) and Theorem 5.1.2 of [2] there exists a maximal NEC-group \(\Lambda\) with this signature. Let \(\Lambda\) be maximal. So by Remark 5.1.1(1) and Theorem 5.1.2 of [2] there exists a maximal NEC-group \(\Lambda\) with this signature. Let \(\theta : \Lambda \to G = Z_2 \times Z_2 \times Z_2 = (x, y, z)\) defined by \(\theta(c_2) = x, \theta(c_{g+1}) = y\) for \(0 \leq j \leq (g-2)/2, \theta(c_g) = x, \theta(c_{g+1}) = z\) and finally \(\theta(c_{g+2}) = zxy\). Then by [2], \(\Gamma = \ker \theta\) is a surface Fuchsian group, and by the Riemann-Hurwitz formula \(X = \mathcal{H}/\Gamma\) is a surface of genus \(g\). Finally its symmetries \(x, y, z\) and \(zxy\) are non-conjugate since \(G = \text{Aut}^\pm(X)\).

Notice that the groups \(G_{\alpha,\alpha+1} = D_n \times Z_2\) and \(G_{\alpha,\alpha+1}\), where \(\alpha\) is even, are isomorphic. In fact, the application
\[\varphi(x) = x, \quad \varphi(y) = y, \quad \varphi(z) = z(xy)^{n/2}x\]
induces an isomorphism \(\varphi : G_{\alpha,\alpha+1} \to D_n \times Z_2\).

In this way, apart from the quantitative result concerning the number of non-conjugate symmetries obtained above, we have obtained also the following qualitative result.

**Corollary 3.4.** Let \(X\) be a Riemann surface of even genus and let \(G\) be a subgroup of \(\text{Aut}^\pm(X)\) generated by non-conjugate symmetries \(\sigma_1, \sigma_2, \sigma_3\) and \(\sigma_4\). Then \(G = D_n \times Z_2\).

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**References**


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