PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 12, December 1998, Pages 3521–3526
S 0002-9939(98)04952-1

SEMIGROUP PROPERTIES OF FACTORS IN THE POLAR DECOMPOSITION OR THE OPERATOR DE-MOIVRE FORMULA

XIMENA CATEPILLÁN AND WACŁAW SZYMAŃSKI

(Communicated by Palle E. T. Jorgensen)

Abstract. We give necessary and sufficient conditions under which the factors in the polar decomposition of a semigroup homomorphism are themselves semigroup homomorphisms.

1. Introduction

For a complex number z its polar form is $z = |z|e^{i\theta}$ with a real number θ. The polar decomposition of a bounded Hilbert space operator $T = V|T|$, where $|T| = (T^*T)^{1/2}$ and V is an appropriate partial isometry, is the non-commutative generalization of the polar form of a complex number. $\mathbb{N}$ is the additive semigroup of natural numbers.

De-Moivre's formula says that $z^n = |z|^n e^{in\theta}$, $n \in \mathbb{N}$.

It is therefore a natural question to ask:

When does $T^n = V^n|T|^n$, $n \in \mathbb{N}$, for an operator T with the polar decomposition $T = |T|V$?

A sufficient condition is, clearly, that T is quasinormal (see definition below). From general results of this paper it will follow that quasinormality is also necessary.

The above question is equivalent to:

When does $|T^n| = |T|^n$ and the partial isometric factor of T^n equal V^n?

Upon a closer examination, this question can be formulated using the semigroup language as follows: The mapping $n \to T^n$, $n \in \mathbb{N}$, is, obviously, a semigroup homomorphism. Hence our question is equivalent to the question: *When are the mappings $n \to |T^n|$ and $n \to$ (the partial isometric factor of T^n) semigroup homomorphisms?*

The only answer in this context has been given by B. Morrel and P. Muhly in [4], Theorem I, who proved that if the operator T is centered (see definition below), then the partial isometric factor of T^n is V^n. Their proof uses the principle of mathematical induction and thus cannot be generalized to any other semigroup.

Before we state the general problem let us introduce some terminology.

$B(H)$ is the algebra of all linear, bounded operators in a Hilbert space H. I denotes the identity operator on H. An operator $V \in B(H)$ is called a *partial isometry* if V^*V is an orthogonal projection, or, equivalently, if $VV^*V = V$. By

Received by the editors January 21, 1997.
1991 *Mathematics Subject Classification.* Primary 47B20, 47D03.

the polar decomposition of $T \in B(H)$ we understand the decomposition $T = V|T|$ with $|T| = (T^*T)^{1/2}$ and $V\colon \overline{R(|T|)} \to \overline{R(T)}$, $V|T|x = Tx,\ x \in H$, $\ker T = \ker V$. $T \in B(H)$ is *quasinormal* if T^*T commutes with T or, equivalently, if the factors in the polar decomposition of T commute. $T \in B(H)$ is called *centered* if the set $\{T^n(T^*)^n, (T^*)^m T^m, n, m \in \mathbb{N}, n, m \neq 0\}$ is commutative. $T \in B(H)$ is *hyponormal* if $T^*T - TT^* \geq 0$.

Let S be a commutative semigroup with 1.

A mapping $\pi\colon S \to B(H)$ is called a *semigroup homomorphism* if $\pi(s+t) = \pi(s)\pi(t),\ s, t \in S$, and $\pi(1) = I$, a *normal homomorphism* if the set $\{\pi(s), \pi(t)^*, s, t \in S\}$ is commutative, a *quasinormal homomorphism* if the set $\{\pi(s)^*\pi(s), \pi(t), s, t \in S\}$ is commutative, a *subnormal homomorphism* if there is a Hilbert space K containing H and a normal homomorphism $\tau\colon S \to B(H)$ such that H is invariant for each $\tau(s)$ and $\tau(s)_{|H} = \pi(s)$, $s \in S$, and a *centered homomorphism* if the set $\{\pi(s)^*\pi(s), \pi(t)\pi(t)^*, s, t \in S\}$ is commutative. All these special kinds of homomorphisms are, clearly, assumed to be semigroup homomorphisms. Clearly, each quasinormal homomorphism is centered.

Let $\pi\colon S \to B(H)$ be a semigroup homomorphism. Let $\pi(s) = \mu(s)\theta(s)$ be the polar decomposition of the operator $\pi(s)$, $s \in S$.

In this paper we will answer the question under which conditions μ and θ are semigroup homomorphisms. This result applied to $S = \mathbb{N}$ will answer our first question, and Theorem 1 applied to the semigroup $[0, \infty)$ will generalize a result of M. Embry in [2], Theorem 6, which she proved under the additional assumption of strong continuity.

2. The results

Let S be a commutative semigroup with 1.

Let $\pi\colon S \to B(H)$ be a semigroup homomorphism, and let $\pi(s) = \theta(s)\mu(s)$ be the polar decomposition of $\pi(s)$, $s \in S$. $E(t) = \theta(t)^*\theta(t)$ is the initial projection for $\theta(t)$ and $F(t) = \theta(t)\theta(t)^*$ is the final projection for $\theta(t)$.

It is known in general that

$$\theta(t)\theta(t)^*\theta(t) = \theta(t),$$

$\theta(t)^*\theta(t)$ is the projection onto $(\ker \pi(t))^\perp = (\ker \mu(t))^\perp = \overline{\operatorname{ran} \mu(t)}$, and

$$\theta(t)^*\theta(t)\mu(t) = \mu(t). \tag{1}$$

Theorem 1. (a) *If π is centered, then θ is a semigroup homomorphism.*

(b) *Assume additionally that for each $s, t \in S$ there exists $r \in S$ such that $s = t + r$ or $t = s + r$. If θ is a semigroup homomorphism, then π is centered.*

Proof. (a) As in the case of one operator, $\mu(s), E(t), F(r)$ commute for all $s, t, r \in S$ (cf. [4]).

$$\mu(t) \text{ and } \theta(r)\mu(r)\theta(r)^* \text{ commute}, \tag{2}$$

because $\mu(t)^2 = \pi(t)^*\pi(t)$ and $\theta(r)\mu(r)^2\theta(r)^* = \pi(r)\pi(r)^*$ commute.

Notice that

$$\theta(r)\mu(r)\theta(r)^* = (\theta(r)\mu(r)^2\theta(r)^*)^{1/2}.$$

Moreover,

$$\theta(t+r)\mu(t+r) = \pi(t+r) = \pi(t)\pi(r) = \theta(t)\mu(t)\theta(r)\mu(r). \tag{3}$$

Since $E(t+r)$ commutes with $\mu(t+r)$ and $E(t+r) \leq E(r)$ because $\operatorname{ran}\mu(t+r) \subset \operatorname{ran}\mu(r)$, it follows that

$$\mu(t+r) = \mu(t+r)E(t+r) = \mu(t+r)E(t+r)E(r) = \mu(t+r)E(r),$$

and by using (3) and (2) we get that

$$\begin{aligned}\theta(t+r)\mu(t+r) &= \theta(t+r)\mu(t+r)E(r) = \theta(t+r)\mu(t+r)\theta(r)^*\theta(r)\\ &= \theta(t)\mu(t)\theta(r)\mu(r)\theta(r)^*\theta(r) = \theta(t)\theta(r)\mu(r)\theta(r)^*\mu(t)\theta(r).\end{aligned}$$

Thus,

$$\theta(t+r)\mu(t+r) = \theta(t)\theta(r)[\mu(r)\theta(r)^*\mu(t)\theta(r)].$$

We claim that the right hand side in this equality is the polar decomposition of $\pi(t+r)$.

First let us prove that $\mu(r)$ and $\theta(r)^*\mu(t)\theta(r)$ commute. Using (1) and (2) we get

$$\begin{aligned}\mu(r)\theta(r)^*\mu(t)\theta(r) &= \theta(r)^*\theta(r)\mu(r)\theta(r)^*\mu(t)\theta(r)\\ &= \theta(r)^*\mu(t)\theta(r)\mu(r)\theta(r)^*\theta(r) = \theta(r)^*\mu(t)\theta(r)\mu(r).\end{aligned}$$

Consequently,

$$\mu(r)\theta(r)^*\mu(t)\theta(r) \geq 0.$$

Moreover,

$$(\mu(r)\theta(r)^*\mu(t)\theta(r))^2 = \pi(t+r)^*\pi(t+r),$$

because

$$\begin{aligned}&\mu(r)\theta(r)^*\mu(t)\theta(r)\mu(r)\theta(r)^*\mu(t)\theta(r)\\ &\quad= \mu(r)\theta(r)^*\mu(t)\theta(r)\theta(r)^*\mu(t)\theta(r)\mu(r) = \pi(r)^*\mu(t)^2\theta(r)\theta(r)^*\theta(r)\mu(r)\\ &\quad= \pi(r)^*\mu(t)^2\theta(r)\mu(r) = \pi(r)^*\pi(t)^*\pi(t)\pi(r) = \pi(t+r)^*\pi(t+r).\end{aligned}$$

Hence,

$$\mu(r)\theta(r)^*\mu(t)\theta(r) = |\pi(t+r)| = \mu(t+r).$$

It remains to be proved that $\theta(t)\theta(r) = \theta(t+r)$. Since $E(t), F(r)$ commute, $\theta(t)\theta(r)$ is a partial isometry. Once it is proved that $\ker\pi(t+r) = \ker\theta(t)\theta(r)$, the uniqueness of the polar decomposition will finish the proof. Since $\ker\pi(t+r) = \ker\mu(t+r)$ and $\mu(t+r) = \mu(r)\theta(r)^*\mu(t)\theta(r)$ as shown above, taking into account that $\ker\mu(s) = \ker\theta(s)$, $s \in S$, we get ($x \in H$)

$$\begin{aligned}&\mu(r)\theta(r)^*\mu(t)\theta(r)x = 0 \Leftrightarrow \theta(r)\theta(r)^*\mu(t)\theta(r)x = 0\\ &\quad\Leftrightarrow \mu(t)\theta(r)\theta(r)^*\theta(r)x = 0 \Leftrightarrow \mu(t)\theta(r)x = 0 \Leftrightarrow \theta(t)\theta(r)x = 0.\end{aligned}$$

(b) Let $r, t \in S$. Then

$$\theta(t+r)\mu(t+r) = \pi(t+r) = \pi(t)\pi(r) = \theta(t)\mu(t)\theta(r)\mu(r).$$

Multiplying on the left by $\theta(t)^*$ and using (1), we get

$$\theta(t)^*\theta(t+r)\mu(t+r) = \mu(t)\theta(r)\mu(r). \tag{4}$$

On the other hand, θ is assumed to be a homomorphism. Therefore,

$$\begin{aligned}\theta(t)^*\theta(t+r) &= \theta(t)^*\theta(t)\theta(r) = E(t)\theta(r)\theta(r)^*\theta(r) = E(t)F(r)\theta(r) = F(r)E(t)\theta(r)\\ &= \theta(r)\theta(r)^*\theta(t)^*\theta(t)\theta(r) = \theta(r)\theta(t+r)^*\theta(t+r).\end{aligned}$$

If we substitute this into (4) and use (1) for $t+r$, we get

$$\theta(r)\mu(t+r) = \mu(t)\theta(r)\mu(r). \tag{5}$$

Now, multiplying on the right by $\theta(r)^*$ we obtain

$$\theta(r)\mu(t+r)\theta(r)^* = \mu(t)[\theta(r)\mu(r)\theta(r)^*].$$

Since the operator on the left hand side, $\mu(t)$, and $\theta(r)\mu(r)\theta(r)^*$ are all self-adjoint, we conclude that $\mu(t)$ commutes with $\theta(r)\mu(r)\theta(r)^*$.

Therefore if we multiply the equality (5) on the left by $\theta(r)^*$ and use the inclusion $\operatorname{ran}\mu(t+r) \subset \operatorname{ran}\mu(r)$, we get

$$\mu(t+r) = \theta(r)^*\theta(r)\mu(t+r) = \theta(r)^*\mu(t)\theta(r)\mu(r).$$

Therefore, since $\mu(t+r)$ is self-adjoint, $\mu(r)$ commutes with $\theta(r)^*\mu(t)\theta(r)$.

Finally,

$$\mu(t+r)\mu(r) = \theta(r)^*\mu(t)\theta(r)\mu(r)^2 = \mu(r)\theta(r)^*\mu(t)\theta(r)\mu(r) = \mu(r)\mu(t+r).$$

Hence, by our additional assumption in (b),

$$\mu(t)\mu(s) = \mu(s)\mu(t), \qquad s,t \in S.$$

Now we are ready to prove the claim, i.e., three commutativity relations. The proofs follow.

First:

$$\pi(s)^*\pi(s)\pi(t)^*\pi(t) = \mu(s)^2\mu(t)^2 = \mu(t)^2\mu(s)^2 = \pi(t)^*\pi(t)\pi(s)^*\pi(s).$$

Second:

$$\begin{aligned}\pi(t)^*\pi(t)\pi(r)\pi(r)^* &= \mu(t)^2\theta(r)\mu(r)^2\theta(r)^* = \mu(t)[\mu(t)\theta(r)\mu(r)]\mu(r)\theta(r)^* \\ &= \mu(t)\theta(r)\mu(t+r)\mu(r)\theta(r)^* = [\mu(t)\theta(r)\mu(r)]\mu(t+r)\theta(r)^* = \theta(r)\mu(t+r)^2\theta(r)^*.\end{aligned}$$

Since the last operator is self-adjoint, $\pi(t)^*\pi(t)$ commutes with $\pi(r)\pi(r)^*$.

Third:

$$\begin{aligned}&\pi(t+r)\pi(t+r)^*\pi(t)\pi(t)^* \\ &\quad= \theta(t+r)\mu(t+r)^2\theta(t+r)^*\theta(t)\mu(t)^2\theta(t) \\ &\quad= \theta(t+r)\mu(t+r)^2\theta(r)^*\theta(t)^*\theta(t)\mu(t)^2\theta(t) \\ &\quad= \theta(t+r)\mu(t+r)^2\theta(r)^*\mu(t)^2\theta(t)^* \\ &\quad= \theta(t)\theta(r)\mu(t+r)\mu(t+r)\theta(r)^*\mu(t)^2\theta(t)^* \\ &\quad= \theta(t)\mu(t)\theta(r)\mu(r)\mu(t+r)\theta(r)^*\mu(t)^2\theta(t)^* \\ &\quad= \theta(t)\mu(t)\theta(r)\mu(t+r)\mu(r)\theta(r)^*\mu(t)^2\theta(t)^* \\ &\quad= \theta(t)\mu(t)\mu(t)\theta(r)\mu(r)\mu(r)\theta(r)^*\mu(t)^2\theta(t)^* \\ &\quad= \theta(t)\mu(t)^2\theta(r)\mu(r)^2\theta(r)^*\mu(t)^2\theta(t)^*.\end{aligned}$$

Since the last operator is self-adjoint, $\pi(t+r)\pi(t+r)^*$ commutes with $\pi(t)\pi(t)^*$. Using the additional assumption in (b) we get that $\pi(s)\pi(s)^*$ commutes with $\pi(t)\pi(t)^*$. $\square$

Our proof of part (b) follows the idea of the proof of Theorem 6 of M. Embry in [2].

In part (b) we made an additional assumption about the semigroup S. Unfortunately, we were not able to prove part (b) without that assumption, which not all semigroups satisfy, e.g., $\mathbb{N} \times \mathbb{N}$ does not. This assumption has a certain order-type flavor, which is, perhaps, essential.

Theorem 2. *Let S be a commutative semigroup with unit and let $\pi\colon S \to B(H)$ be a semigroup homomorphism. π is a quasinormal homomorphism if and only if μ is a semigroup homomorphism.*

Proof. Suppose π is a quasinormal homomorphism. Notice that $\theta(t), \mu(t), \theta(s)$, and $\mu(s)$ commute, $s, t \in S$.

Thus $\pi(s+t) = \pi(s)\pi(t) = \theta(s)\mu(s)\theta(t)\mu(t) = \theta(s)\theta(t)\mu(s)\mu(t)$. Now, since π is quasinormal, it is centered and therefore θ is also a homomorphism by Theorem 1. Hence, $\pi(t+s) = \theta(s+t)\mu(s)\mu(t)$ and by the uniqueness of the polar decomposition we get that $\mu(s+t) = \mu(s)\mu(t)$. This part of the proof can essentially be found in A. Lubin [3], proof of Theorem 4.5.

Suppose now that μ is a semigroup homomorphism. To prove that π is quasinormal we use one of the equivalent conditions for quasinormality from Theorem 3.10 in W. Szymański [5], namely, π is a quasinormal homomorphism if and only if each operator $\pi(s)$, $s \in S$, is quasinormal and π is a subnormal homomorphism.

(i) To prove that π is subnormal we use the following criterion to characterize subnormal homomorphisms (W. Szymański [5]):

(EL) π is subnormal if and only if

$\Sigma(\pi(s+t)f(t), \pi(s+t)f(s)) \geq 0$ for all functions $f\colon S \to H$ with finite support.

Let f be such a function. Then

$$\begin{aligned}
&\Sigma(\pi(s+t)f(t), \pi(s+t)f(s)) \\
&\quad = \Sigma(\theta(s+t)\mu(s+t)f(t), \theta(s+t)\mu(s+t)f(s)) \\
&\quad = \Sigma(\theta(s+t)^*\theta(s+t)\mu(s+t)f(t), \mu(s+t)f(s)) \\
&\quad = \Sigma(\mu(s+t)f(t), \mu(s+t)f(s)) = \Sigma(\mu(s+t)\mu(s+t)f(t), f(s)) \\
&\quad = \Sigma(\mu(s+s+t+t)f(t), f(s)) = \Sigma(\mu(s+s)\mu(t+t)f(t), f(s)) \\
&\quad = \Sigma(\mu(s)^2\mu(t)^2 f(t), f(s)) = \Sigma(\mu(t)^2 f(t), \mu(s)^2 f(s)) = \|\Sigma\mu(s)^2 f(s)\|^2 \geq 0.
\end{aligned}$$

Thus π is subnormal.

(ii) Now we only need to prove that each $\pi(s)$, $s \in S$, is quasinormal. Each $\pi(s)$, $s \in S$, is subnormal, because π is a subnormal homomorphism, as shown in (i) above. Hence each $\pi(s)$ is hyponormal, $s \in S$. A simple observation made by M. Embry in [1], page 63, that $T \in B(H)$ is quasinormal if and only if T is hyponormal and $|T^2| = |T|^2$, completes the proof. Indeed, let $T = \pi(s)$, $s \in S$. Since μ is a semigroup homomorphism, it follows that $|T^2| = |\pi(2s)| = \mu(2s) = \mu(s)^2 = |T|^2$. □

References

[1] M. Embry, *A generalization of the Halmos-Bram criterion for subnormality*, Acta. Math. (Szeged) **35** (1973), 61–64. MR **48:**6994

[2] ———, *The partially isometric factor of a semigroup*, Indiana Univ. Math. Journal **32** (1983), 893–901. MR **86h:**47059

[3] A. Lubin, *Weighted shifts and commuting normal extensions*, J. Austral. Math. Soc. Ser. A **27** (1979), 17–26. MR **80j:**47030

[4] B. Morrel and P. Muhly, *Centered operators*, Studia Mathematica **51** (1974), 251–263. MR **50:**8132

[5] W. Szymański, *Dilations and subnormality*, Proc. Amer. Math. Soc. **101** (1987), 251–259. MR **88k:**47062

Department of Mathematics, Millersville University, Millersville, Pennsylvania 17551

E-mail address: xcatepil@marauder.millersv.edu

Department of Mathematics, West Chester University, West Chester, Pennsylvania 19383

E-mail address: wszymans@wcupa.edu