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EIGENVALUES OF A STURM-LIOUVILLE PROBLEM AND INEQUALITIES OF LYAPUNOV TYPE

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Abstract. We consider the eigenvalue problem $u'' + \lambda u + p(x)u = 0$ in $(0, \pi)$, $u(0) = u(\pi) = 0$, where $p \in L^1(0, \pi)$ keeps a fixed sign and $\|p\|_{L^1} > 0$, and we obtain some lower and upper bounds for $\|p\|_{L^1}$ in terms of its nonnegative eigenvalues λ. Two typical results are: (1) $\|p\|_{L^1} > \sqrt{\lambda}\,|\sin\sqrt{\lambda}\,\pi|$ if $\lambda > 1$ and is not the square of a positive integer; (2) $\|p\|_{L^1} \leq 16/\pi$ if $\lambda = 0$ is the smallest eigenvalue.

1. Introduction

We consider the eigenvalue problem

$$u'' + \lambda u + p(x)u = 0 \quad \text{in } (0, \pi), \quad u(0) = u(\pi) = 0, \tag{1}$$

where $p \in L^1(0, \pi)$. It follows from the classical inequality of Lyapunov that if $\lambda = 0$ is an eigenvalue of the problem (1), then necessarily

$$\|p\|_{L^1} > \pi/4. \tag{2}$$

Over the years there have appeared a number of improvements and extensions of this interesting result (see, e.g. [1], [2]). The purpose of this paper is to extend the original Lyapunov inequality in yet another direction. We treat the problem when λ is any nonnegative eigenvalue of (1).

We assume throughout that

$$p \in L^1(0, \pi),\ \|p\|_{L^1} > 0 \text{ and either } p \geq 0 \text{ or } p \leq 0 \text{ a.e. on } (0, \pi). \tag{3}$$

Our main results are given in §2 in which some lower bounds for $\|p\|_{L^1}$ are obtained in terms of the nonnegative eigenvalues of (1). In §3 we derive some upper bounds for $\|p\|_{L^1}$ which ensure an eigenvalue $\lambda \in [0, 1]$ of (1) to be the smallest eigenvalue. For their proofs, we use basic properties of some boundary value problems and their Green functions (see, e.g. [7]). It was observed in [5] that (2) could be proved using the Green function method. We refer to [6, Chap. 2] for other interesting results relating to the eigenvalue problem (1). This work is motivated by a study on the solvability of some nonlinear boundary value problems (see [3], [4]).

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2. Inequalities of Lyapunov type

In this section some inequalities of Lyapunov type are obtained. We always denote by n a generic positive integer and consider the cases $\lambda \neq n^2$ and $\lambda = n^2$ separately in Theorems 1 and 2 below.

We suppose first that $\lambda \neq n^2$. We recall that for any $h \in L^1(0,\pi)$, the boundary value problem

$$u'' + \lambda u + h(x) = 0 \quad \text{in } (0,\pi), \quad u(0) = u(\pi) = 0 \tag{4}$$

has a unique solution u. More precisely, if $\lambda > 0$, we have

$$u(x) = \int_0^\pi G(x,\xi,\lambda) h(\xi)\, d\xi,$$

where

$$G(x,\xi,\lambda) = \begin{cases} \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi} \sin\sqrt{\lambda}\,\xi \sin\sqrt{\lambda}\,(\pi - x) & \text{if } 0 \leq \xi \leq x, \\ \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi} \sin\sqrt{\lambda}\,x \sin\sqrt{\lambda}\,(\pi - \xi) & \text{if } x \leq \xi \leq \pi \end{cases}$$

is the Green function for the problem (4). It follows by a simple calculation that

$$|G(x,\xi,\lambda)| < \begin{cases} \frac{1}{2\sqrt{\lambda}} \tan \frac{\sqrt{\lambda}}{2}\pi & \text{if } 0 < \lambda < 1, \\ \frac{1}{\sqrt{\lambda}} |\csc\sqrt{\lambda}\,\pi| & \text{if } \lambda > 1 \end{cases} \tag{5}$$

holds a.e. on the square $0 \leq \xi,\ x \leq \pi$.

Theorem 1. *If $\lambda > 0$ is an eigenvalue of (1), $\lambda \neq n^2$, then*

$$\|p\|_{L^1} > \begin{cases} 2\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{2}\pi & \text{if } 0 < \lambda < 1, \\ \sqrt{\lambda}\,|\sin\sqrt{\lambda}\,\pi| & \text{if } \lambda > 1. \end{cases} \tag{6}$$

Proof. We assume that $p \geq 0$ a.e. on $(0,\pi)$. The other case can be proved similarly. Let u be an eigenfunction of (1) corresponding to λ. Since u is a nontrivial solution to (4) with $h(x) = p(x)u$, it follows from the remark above that

$$u(x) = \int_0^\pi G(x,\xi,\lambda) p(\xi) u(\xi)\, d\xi. \tag{7}$$

Taking the inner product in $L^2(0,\pi)$ of (7) with $p(x)u$, we have

$$\begin{aligned} \int_0^\pi p(x)|u(x)|^2\, dx &= \int_0^\pi \int_0^\pi G(x,\xi,\ \lambda) p(\xi) u(\xi) p(x) u(x)\, d\xi\, dx \\ &< m(\lambda)^{-1} \left(\int_0^\pi p(x)|u(x)|\, dx \right)^2, \end{aligned}$$

where $m(\lambda)$ is the function defined by the right-hand side of (6), which is reciprocal to that of the right-hand side of (5). Hence

$$\begin{aligned} \int_0^\pi p(x)|u(x)|\, dx &\leq \left(\int_0^\pi p(x)|u(x)|^2\, dx \right)^{\frac{1}{2}} \left(\int_0^\pi p(x)\, dx \right)^{\frac{1}{2}} \\ &< m(\lambda)^{-\frac{1}{2}} \left(\int_0^\pi p(x)|u(x)|\, dx \right) \left(\int_0^\pi p(x)\, dx \right)^{\frac{1}{2}}. \end{aligned} \tag{8}$$

By hypothesis $\int_0^\pi p(x)|u(x)|\, dx \neq 0$, and so the result follows. ∎

Obviously the inequality (2) follows from a similar line of arguments using instead the corresponding Green function

$$G(x,\xi,\,0)=\begin{cases}\frac{1}{\pi}\xi(\pi-x) & \text{if } 0\le\xi\le x,\\ \frac{1}{\pi}x(\pi-\xi) & \text{if } x\le\xi\le\pi.\end{cases}$$

As $G(x,\xi,\lambda)$ is continuous at $\lambda=0$, we have $\lim_{\lambda\to 0^+} m(\lambda)=4/\pi$ as expected.

We now suppose that $\lambda=n^2$. We recall that for any $h\in L^1(0,\pi)$, the boundary value problem

$$u''+n^2u+h(x)=0 \quad \text{in } (0,\pi), \quad u(0)=u(\pi)=0 \tag{9}$$

has a solution if and only if $\int_0^\pi h(x)\sin nx\,dx=0$. If this is the case, there exists a unique solution u to (9) such that $\int_0^\pi u(x)\sin nx\,dx=0$. More precisely, we have

$$u(x)=\int_0^\pi G(x,\xi,n^2)h(\xi)\,d\xi,$$

where

$$G(x,\xi,n^2)=-\xi\cos n\xi(\sin nx)/n\pi+\begin{cases}\sin n\xi(\cos nx)/n & \text{if } 0\le\xi\le x,\\ \cos n\xi(\sin nx)/n & \text{if } x\le\xi\le\pi\end{cases}$$

is the Green function for the problem (9).

Theorem 2. *If $\lambda=n^2$ is an eigenvalue of (1), then*

$$\|p\|_{L^1}>2n.$$

Proof. We assume that $p\ge 0$ a.e. on $(0,\pi)$. The other case can be proved similarly. Let u be an eigenvalue of (1) corresponding to $\lambda=n^2$. Since u is a nontrivial solution to (9) with $h(x)=p(x)u$, it follows from the remark above that

$$u(x)=(2/\pi)\left(\int_0^\pi u(\xi)\sin n\xi\,d\xi\right)\sin nx+\int_0^\pi G(x,\xi,\,n^2)p(\xi)u(\xi)\,d\xi. \tag{10}$$

Moreover,

$$\int_0^\pi p(x)u(x)\sin nx\,dx=0 \tag{11}$$

and so taking the inner product in $L^2(0,\pi)$ of (10) with $p(x)u$, we have

$$\begin{aligned}\int_0^\pi p(x)|u(x)|^2\,dx&=\int_0^\pi\int_0^\pi G(x,\xi,\,n^2)p(\xi)u(\xi)p(x)u(x)\,d\xi dx\\ &=(1/n)\left[\int_0^\pi\int_0^x \sin n\xi(\cos nx)p(\xi)u(\xi)p(x)u(x)\,d\xi dx\right.\\ &\quad\left.+\int_0^\pi\int_x^\pi \cos n\xi(\sin nx)p(\xi)u(\xi)p(x)u(x)\,d\xi dx\right].\end{aligned}$$

It follows by (11) again that

$$\begin{aligned}&\left[\int_0^\pi\int_0^x+\int_0^\pi\int_x^\pi \cos n\xi(\sin nx)p(\xi)u(\xi)p(x)u(x)\,d\xi dx\right]\\ &=\int_0^\pi\int_0^\pi \cos n\xi(\sin nx)p(\xi)u(\xi)p(x)u(x)\,d\xi dx=0\end{aligned}$$

and hence

$$\int_0^\pi p(x)|u(x)|^2\,dx = (1/n)\int_0^\pi\int_0^x \sin n(\xi - x)p(\xi)u(\xi)p(x)u(x)\,d\xi dx$$
$$< (1/2n)\left(\int_0^\pi p(x)|u(x)|\,dx\right)^2.$$

We obtain as in (8) that

$$\int_0^\pi p(x)|u(x)|\,dx < (1/\sqrt{2n})\left(\int_0^\pi p(x)|u(x)|\,dx\right)\left(\int_0^\pi p(x)\,dx\right)^{\frac{1}{2}}.$$

By hypothesis $\int_0^\pi p(x)|u(x)|\,dx \neq 0$, and so the result follows. □

We note that the assumption $\|p\|_{L^1} > 0$ is essential for Theorem 2 to hold. Clearly $u = \sin nx$ is an eigenfunction of (1) corresponding to n^2 when $p = 0$ a.e. on $(0, \pi)$.

3. Upper bounds for $\|p\|_{L^1}$

In this section we obtain some upper bounds of $\|p\|_{L^1}$ which are sufficient for an eigenvalue $\lambda \in [0, 1]$ of (1) to be the smallest eigenvalue. We recall that if λ is the smallest eigenvalue of (1), the eigenfunction u corresponding to λ has no zero in $(0, \pi)$. This is the property our proof is based on.

Theorem 3. *Let $0 \le \lambda \le 1$. If λ is an eigenvalue of (1) and*

$$\|p\|_{L^1} \le \begin{cases} \frac{16}{\pi} & \text{if } \lambda = 0, \\ 4\sqrt{\lambda}\cot\frac{\sqrt{\lambda}}{4}\pi & \text{if } 0 < \lambda \le 1, \end{cases} \tag{12}$$

then λ is the smallest eigenvalue.

Proof. Let u be an eigenfunction of (1) corresponding to λ. We suppose on the contrary that u has a zero $l \in (0, \pi)$ and define

$$v(x) = u(\frac{l}{\pi}x), \qquad w(x) = u(\frac{\pi - l}{\pi}x + l).$$

Then v and w are nontrivial solutions of (1) with λ replaced by $(\frac{l}{\pi})^2\lambda$ and $(\frac{\pi-l}{\pi})^2\lambda$, $p(x)$ replaced by $(\frac{l}{\pi})^2 p(\frac{l}{\pi}x)$ and $(\frac{\pi-l}{\pi})^2 p(\frac{\pi-l}{\pi}x + l)$, respectively.

If $0 < \lambda \le 1$, we apply Theorem 1 to obtain

$$\int_0^l |p(x)|\,dx > 2\sqrt{\lambda}\cot\frac{l}{2}\sqrt{\lambda}; \qquad \int_l^\pi |p(x)|\,dx > 2\sqrt{\lambda}\cot\frac{\pi - l}{2}\sqrt{\lambda}.$$

It follows by a simple calculation that

$$\int_0^\pi |p(x)|\,dx > 2\sqrt{\lambda}\left[\cot\frac{l}{2}\sqrt{\lambda} + \cot\frac{\pi - l}{2}\sqrt{\lambda}\right]$$
$$\ge 4\sqrt{\lambda}\cot\frac{\sqrt{\lambda}}{4}\pi$$

which contradicts (12).

If $\lambda = 0$, we apply (2) to obtain

$$\int_0^\pi |p(x)|\,dx = \left[\int_0^l + \int_l^\pi |p(x)|\,dx\right]$$
$$> 4\left(\frac{1}{l} + \frac{1}{\pi - l}\right) \geq \frac{16}{\pi}$$

which again contradicts (12). This completes the proof of the theorem. □

References

1. L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, Springer Verlag, Berlin, 1971. MR **50:**2582
2. A.M. Fink and D.F. St. Mary, *On an inequality of Nehari*, Proc. Amer. Math. Soc. **21** (1969), 640–642. MR **39:**1737
3. C.W. Ha and C.C. Kuo, *On the solvability of a two point boundary value problem at resonance*, Topol. Methods Nonlinear Anal. **1** (1993), 295–302. MR **94g:**34028
4. C.W. Ha and C.C. Kuo, *On the solvability of a two point boundary value problem at resonance, II*, Topol. Methods Nonlinear Anal. to appear.
5. Z. Nehari, *On the zeros of solutions of second order linear differential equations*, Amer. J. Math. **76** (1954), 689–697. MR **16:**131f
6. J. Pöschel and E. Trubowitz, *Inverse spectral theory*, Academic Press, Orlando, FL, 1987. MR **89b:**34061
7. I. Stakgold, *Green's functions and boundary value problems*, John Wiley *and* Sons, New York, 1979. MR **80k:**35002

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