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ON ONE SET OF ORTHOGONAL HARMONIC POLYNOMIALS

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Abstract. A new basis of harmonic polynomials is given. Proposed polynomials are orthogonal on the unit sphere and each term of this basis consists of monomials not present in the others.

Introduction

For the investigation of harmonic polynomials a scalar product for homogeneous polynomials of degree m in the form $\langle P_m(x), Q_m(x)\rangle = P_m(D)Q_m(x)$ was introduced in [1]—one of the basic works on harmonic analysis— where the operator $P_m(D)$ is obtained from the polynomial $P_m(x)$ by replacing each variable x_i on the differential operator $\partial/\partial x_i$. If we denote a set of all polynomials over C by $\mathcal{P}$, then that scalar product can be extended on $\mathcal{P}$ in the following way: $\langle P(x), Q(x)\rangle = P(D)Q(x)_{|x=0}$. This idea proved to be very successful for the investigation of polynomial solutions to systems of PDE with constant coefficients (see for instance [2], [3]). In the present work we shall consider a full set of harmonic polynomials which are furthermore orthogonal both in $L_2(\partial S_n)$ (S_n is a unit ball in R^n) and in $\mathcal{P}$, and we shall give one interesting property of theirs (Corollary 2).

1. System of harmonic polynomials $G_{(\nu)}(x)$

Let $k, s \in \widetilde{N}$ ($\widetilde{N} \equiv N \cup \{0\}$) and $n \in N$ but $n > 1$. Consider the polynomials

$$G_k^s(x_{(n)}) = \sum_{i=0}^{[k/2]} (-1)^i \frac{|x_{(n-1)}|^{2i} x_n^{k-2i,!}}{(2,2)_i(n-1+2s,2)_i},$$

where $(a,b)_i = a(a+b)\dots(a+ib-b)$ (with the convention $(a,b)_0 = 1$), $x^{k,!} = x^k/k!$, and $x_{(n)} = (x_1, \dots, x_n)$. We shall call them G-polynomials of degree k, order s, and kind n.

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If $n = 2$, then there are only two linearly independent homogeneous harmonic polynomials of degree k $(k > 0)$ and we can write them in the form

$$H_k^s(x_{(2)}) = \sum_{i=0}^{[(k-s)/2]} (-1)^i x_1^{2i+s,!} x_2^{k-2i-s,!}$$

for $k \in N$, $s = 0, 1$, and with the convention $H_0^0 \equiv 1$. It is not difficult to verify that $H_k^s(x_{(2)}) = G_{k-s}^s(x_{(2)}) x_1^s$.

Theorem 1. *Let $n \geq 2$. The polynomials*

$$G_{(\nu)}(x_{(n)}) = G_{\nu_1-\nu_2}^{\nu_2}(x_{(n)}) \cdots G_{\nu_{n-1}-\nu_n}^{\nu_n}(x_{(2)}) x_1^{\nu_n}, \tag{1}$$

where $\nu \in \widetilde{N}^n$, $\nu_1 \geq \cdots \geq \nu_n$, and $\nu_n = 0, 1$, make up a basis among homogeneous harmonic polynomials of degree ν_1.

Remark. Polynomials $G_{(\nu)}$ can be determined also by the recurrence relation

$$\begin{array}{ll} G_{(\nu)}(x) = H_{\nu_{n-1}}^{\nu_n}(x), & n = 2, \\ G_{(\nu)}(x_{(n)}) = G_{\nu_1-\nu_2}^{\nu_2}(x_{(n)}) G_{(\bar{\nu})}(x_{(n-1)}), & n > 2, \end{array}$$

where $\bar{\nu} = (\nu_2, \ldots, \nu_n)$.

Proof. At first, for convenience, denote $k = \nu_1$ and $s = \nu_2$. We shall employ induction on the dimension n. If $n = 2$, then $G_{(\nu)}(x_{(n)}) = G_{k-s}^s(x_{(2)}) x_1^s = H_k^s(x_{(2)})$ and the theorem's statement is obviously true.

Let $n \geq 3$. Consider a set of homogeneous polynomials of degree k having the form $v_k(x_{(n)}) = u_s(x_{(n-1)}) G_{k-s}^s(x_{(n)})$, where k is fixed, $s = 0, \ldots, k$, and the polynomials $u_s(x_{(n-1)})$ make up a basis among homogeneous harmonic (of $n-1$ variables) polynomials of degree s, which according to the induction hypothesis can be taken in the form (1) for $n = n - 1$.

Let the vector μ correspond to some polynomial $u_s(x_{(n-1)})$. If we denote $\nu = (k, \mu_1, \ldots, \mu_{n-1})$, then the polynomial $v_k(x_{(n)})$ is of the form (1).

In a straightforward way from the Laplace equation we can verify that for $s \leq k$ the polynomials $v_k(x_{(n)})$ are harmonic.

It is not hard to see that the polynomials of the form $v_k(x_{(n)})$ are linearly independent. Indeed, if s is fixed they are linearly independent because of the linear independence of $u_s(x_{(n-1)})$, if $s = 0, \ldots, k$ they have different degrees with respect to x_n and therefore are linearly independent also.

Finally, we need to count the number of $v_k(x_{(n)})$. It is known [1] that the number of $u_s(x_{(n-1)})$ for $s > 1$ is

$$h_s^{n-1} = \binom{s+n-2}{n-2} - \binom{s+n-4}{n-2}$$

and $h_1^{n-1} = n - 1$, $h_0^{n-1} = 1$. Therefore the number of $v_k(x_{(n)})$ will be

$$\sum_{s=2}^{k} \left(\binom{s+n-2}{n-2} - \binom{s+n-4}{n-2} \right) + n.$$

After some transformations this number becomes equal to h_k^n. This ends the proof. □

Corollary 1. *Let $\mathcal{G}_{n-1}$ be a basis in harmonic polynomials of $n-1$ variables. Then*

$$\mathcal{G}_n = \{G_k^s(x_{(n)})u_s(x_{(n-1)}) \mid k \in \widetilde{N}, u_s \in \mathcal{G}_{n-1}\}$$

is a basis in harmonic polynomials of n variables.

We can easily extract the proof of this statement from the previous theorem.

In what follows, if no confusion will result, instead of $x_{(n)}$ and $x_{(n-1)}$ we shall write x and $\tilde{x}$ respectively. Denote a square of ∂S_n by ω_n and the norm of $u \in L_2(\partial S_n)$ by $\|u\|$.

Lemma 1. *Let $f \in C(\partial S_n)$ be taken in the form $f(x) = \varphi(|\tilde{x}|, x_n)P_k(\tilde{x})$, where $P_k(\tilde{x})$ is a homogeneous polynomial of degree k and $\varphi \in C(\partial S_n)$. Then*

$$\int_{|x|=1} f(x)\,dx = \frac{1}{\omega_{n-1}} \int_{|x|=1} \varphi(|\tilde{x}|, x_n)\left(1 - x_n^2\right)^{k/2}\,dx \int_{|\tilde{x}|=1} P_k(\tilde{x})\,d\tilde{x}. \tag{2}$$

Proof. We can lightly verify that for $f \in C(\partial S_n)$ the following equality holds:

$$\int_{|x|=1} f(x)\,dx = \int_{|\tilde{x}|<1} \left[f\left(\tilde{x}, \sqrt{1-|\tilde{x}|^2}\right) + f\left(\tilde{x}, -\sqrt{1-|\tilde{x}|^2}\right)\right]\left(1-|\tilde{x}|^2\right)^{-1/2}\,d\tilde{x}.$$

Further, making some transformations we get

$$\int_{|x|=1} f(x)\,dx = \int_0^1 \frac{\tau^{n-2}}{\sqrt{1-\tau^2}}\,d\tau \int_{|\tilde{x}|=1} \left[f\left(\tau\tilde{x}, \sqrt{1-\tau^2}\right) + f\left(\tau\tilde{x}, -\sqrt{1-\tau^2}\right)\right]\,d\tilde{x}.$$

If we make the substitution $t = \sqrt{1-\tau^2}$, then this yields

$$\int_{|x|=1} f(x)\,dx = \int_0^1 (1-t^2)^{(n-3)/2}\,dt \cdot \int_{|\tilde{x}|=1} \left[f\left(\sqrt{1-t^2}\tilde{x}, t\right) + f\left(\sqrt{1-t^2}\tilde{x}, -t\right)\right]\,d\tilde{x}.$$

Dividing the integral on the right into two parts and substituting t for $-t$ in the second integral we get

$$\int_{|x|=1} f(x)\,dx = \int_{-1}^1 (1-t^2)^{(n-3)/2}\,dt \int_{|\tilde{x}|=1} f\left(\sqrt{1-t^2}\tilde{x}, t\right)\,d\tilde{x}. \tag{3}$$

Now it is time to use the form of $f(x)$. We have

$$\int_{|x|=1} f(x)\,dx = \int_{-1}^1 \left(\sqrt{1-t^2}\right)^{n+k-3} \varphi\left(\sqrt{1-t^2}, t\right)\,dt \int_{|\tilde{x}|=1} P_k(\tilde{x})\,d\tilde{x}.$$

If we multiply this equality by

$$\omega_{n-1} = \int_{|\tilde{x}|=1} |\tilde{x}|^k\,d\tilde{x}$$

and use again the equality (3) with the function $f(x) = \varphi\left(|\tilde{x}|, x_n\right)|\tilde{x}|^k$, then we get

$$\omega_{n-1} \int_{|x|=1} f(x)\,dx = \int_{|x|=1} \varphi(|\tilde{x}|, x_n)\left(1-x_n^2\right)^{k/2}\,dx \int_{|\tilde{x}|=1} P_k(\tilde{x})\,d\tilde{x},$$

which implies (2). The proof is complete. □

Lemma 2. *G-polynomials of the same order s and kind n are orthogonal with weight $\rho_n^s(x) = (1-x_n^2)^s$ on ∂S_n.*

Proof. Let $G_p^s(x)$ and $G_q^s(x)$ $(p \neq q)$ be two arbitrary G-polynomials of the same kind n, and let $u_s(\tilde{x})$ be any homogeneous harmonic polynomial of degree s. If we denote $u_{p+s}(x) = G_p^s(x)u_s(\tilde{x})$ and $u_{q+s}(x) = G_q^s(x)u_s(\tilde{x})$ respectively, then the polynomials $u_{p+s}(x)$ and $u_{q+s}(x)$ are, according to Corollary 1, harmonic. Let us take advantage of Lemma 1 for the function $\varphi(x) = G_p^s(x)G_q^s(x)$ and the polynomial $P_k(\tilde{x}) = u_s^2(\tilde{x})$. This is possible because $G_k^s(x) = G_k^s(|\tilde{x}|, x_n)$ and Lemma 1's condition is fulfilled. We have

$$\int_{|x|=1} u_{p+s}(x)u_{q+s}(x)\,dx = \frac{1}{\omega_{n-1}}\int_{|x|=1} \rho_n^s(x)G_p^s(x)G_q^s(x)\,dx \int_{|\tilde{x}|=1} u_s^2(\tilde{x})\,d\tilde{x}.$$

Since the left-hand side of this equality vanishes $(p+s \neq q+s)$ [1] and $\|u_s\| \neq 0$ we arrive at the desired result. The proof is complete. □

Now we are able to establish an orthogonality of the system $\{G_{(\nu)}\}$ on ∂S_n.

Theorem 2. *Polynomials $G_{(\nu)}(x)$ for any distinct vectors $\nu \in \widetilde{N}^n$ satisfying the condition $\nu_1 \geq \cdots \geq \nu_n$ $(\nu_n = 0, 1)$ are orthogonal on ∂S_n.*

Proof. In just the same way as in Theorem 1 we employ the induction on the dimension n. Let $n = 2$ and $\nu = (m, s)$. For $x \in \partial S_2$ the polynomials $G_{(\nu)}(x) = H_m^s(x)$ can be represented in the form

$$H_m^0(x) = \frac{1}{m!}\cos(m \arccos x_2), \quad H_m^1(x) = \frac{1}{m!}\sin(m \arccos x_2),$$

where $x_2 \in [-1, 1]$ and therefore are orthogonal on ∂S_2.

Let $n \geq 3$ and ν, μ be two unequal vectors from $\widetilde{N}^n$ satisfying the theorem's condition. Let us take advantage of Lemma 1 keeping in mind the Remark to Theorem 1. We have

$$\int_{|x|=1} G_{(\nu)}(x)G_{(\mu)}(x)\,dx =$$

(4)

$$\frac{1}{\omega_{n-1}}\int_{|x|=1} G_{\nu_1-\nu_2}^{\nu_2}(x)G_{\mu_1-\mu_2}^{\mu_2}(x)\left(1-x_n^2\right)^{(\nu_2+\mu_2)/2}\,dx \int_{|\tilde{x}|=1} G_{(\bar{\nu})}(\tilde{x})G_{(\bar{\mu})}(\tilde{x})\,d\tilde{x}.$$

There are only two possibilities: either $\nu_1 \neq \mu_1$, $\bar{\nu} = \bar{\mu}$ or $\nu_1 = \mu_1$, $\bar{\nu} \neq \bar{\mu}$. In the first case the right-hand side of (4) vanishes by virtue of Lemma 2 ($\nu_1 \neq \mu_1$, $\nu_2 = \mu_2 \Rightarrow \nu_1 - \nu_2 \neq \mu_1 - \mu_2$) and in the second case, by virtue of the induction hypothesis. The proof is complete. □

Theorem 3. *Polynomials $G_{(\nu)}(x)$ for any distinct vectors $\nu \in \widetilde{N}^n$ satisfying the condition $\nu_1 \geq \cdots \geq \nu_n$ $(\nu_n = 0, 1)$ are orthogonal in $\mathcal{P}$.*

Proof. We need to check the correctness of the following statement: $\nu \neq \mu \Rightarrow \langle G_{(\nu)}, G_{(\mu)}\rangle = 0$. To do this we are going to employ again the induction on the dimension n. We shall consider the case $\nu_1 = \mu_1$ because if $\nu_1 \neq \mu_1$, then the polynomials $G_{(\nu)}(x)$ and $G_{(\mu)}(x)$ are orthogonal as having different degree.

Let $n = 2$. The polynomials $G_{(\nu)}(x)$ and $G_{(\mu)}(x)$ are orthogonal because they have different evenness with respect to x_1: if $\nu_2 = 0$, then $G_{(\nu)}(x)$ is even, and if $\nu_2 = 1$, it is odd.

Let $n > 2$. Consider the case $\nu_2 < \mu_2$. Determine G-polynomials for $k \in Z\backslash\widetilde{N}$ as $G_k^s \equiv 0$ and denote $\widetilde{\Delta} = \Delta - \partial^2/\partial x_n^2$. It is not difficult to verify that

$$\frac{\partial}{\partial x_n}\, G_{(\mu)}(x) = G_{(\mu_-)}(x), \quad \widetilde{\Delta} G_{(\mu)}(x) = -G_{(\mu_=)}(x),$$

where $\mu_- = (\mu_1 - 1, \mu_2 \dots, \mu_n)$ and $\mu_= = (\mu_-)_-$. Therefore using the Remark to Theorem 1 we can get

$$G_{(\nu)}(D)G_{(\mu)}(x) = CG_{(\bar{\nu})}(\widetilde{D})\left[G^{\mu_2}_{(\mu_1-\mu_2)-(\nu_1-\nu_2)}(x)G_{(\bar{\nu})}(\tilde{x})\right],$$

where

$$C = \sum_{i=0}^{[\nu_1/2]} \frac{1}{(2,2)_i(n-1+2\nu_2,2)_i(\nu_1-2i)!}.$$

Since $\mu_1 - \mu_2 - \nu_1 + \nu_2 = \nu_2 - \mu_2 < 0$ we can conclude that $G_{(\nu)}(x)$ and $G_{(\mu)}(x)$ are orthogonal. The same is true for $\nu_2 > \mu_2$ also. Consider the last case $\nu_2 = \mu_2$. Using the previous equality we obtain $\langle G_{(\nu)}, G_{(\mu)}\rangle = C\langle G_{(\bar{\nu})}, G_{(\bar{\mu})}\rangle$. Since $\nu_1 = \mu_1$ and $\nu \neq \mu$ we have $\bar{\nu} \neq \bar{\mu}$. Hence, by the induction hypothesis $G_{(\nu)}(x)$ and $G_{(\mu)}(x)$ are orthogonal. The proof is complete. ☐

2. Property of the polynomials $G_{(\nu)}(x)$

Consider the polynomials $\overline{G}_{(\nu)}(x)$ to be obtained from $G_{(\nu)}(x)$ by normalization in $L_2(\partial S_n)$. In what follows without stipulation we shall use ν as a vector from $\widetilde{N}^n$ satisfying the condition $\nu_1 \geq \cdots \geq \nu_n$ $(\nu_n = 0, 1)$. For $n > 2$ denote

$$E(x,\xi) = \frac{1}{(n-2)\omega_n}\, |x-\xi|^{2-n}.$$

Lemma 3. *If $|x| < |\xi|$, then*

$$E(x,\xi) = \sum_{\nu} \frac{|\xi|^{-(2\nu_1+n-2)}}{2\nu_1+n-2}\, \overline{G}_{(\nu)}(x)\overline{G}_{(\nu)}(\xi). \tag{5}$$

Proof. Let $C_k^l[t]$ be Gegenbauer's polynomial [4]. So if $|\xi| = 1$ and $|x| < 1$ we have

$$|x-\xi|^{2-n} = \sum_{k=0}^{\infty} C_k^{n/2-1}\left[\left(\frac{x}{|x|}, \xi\right)\right]|x|^k. \tag{6}$$

According to [4] for any $k \in \widetilde{N}$ and $\xi, \eta \in \partial S_n$ we can get the equality

$$\frac{1}{\omega_n}\, \frac{C_k^{n/2-1}[(\xi,\eta)]}{C_k^{n/2-1}[1]} = \frac{1}{h_k^n} \sum_{\nu_1=k} \overline{G}_{(\nu)}(\xi)\overline{G}_{(\nu)}(\eta),$$

where h_k^n is defined in Theorem 1, summation is taken over all ν such that $\nu_1 = k$, and $C_k^{n/2-1}[1] = (k+n-3)!/(k!(n-3)!)$. After using this expression in (6) we obtain

$$\frac{1}{\omega_n}\, |x-\xi|^{2-n} = \sum_{k=0}^{\infty} \frac{n-2}{2k+n-2} \sum_{\nu_1=k} \overline{G}_{(\nu)}(x)\overline{G}_{(\nu)}(\xi). \tag{7}$$

If now observe that $E(x,\xi) = |\xi|^{2-n}E(x/|\xi|, \xi/|\xi|)$, where $|x/|\xi|| < 1$, then (7) can be easily transformed to the form (5). The proof is complete. ☐

Lemma 4. *Let $\xi,\eta\in\partial S_n$ and $k\in\widetilde{N}$. Then*

$$\frac{1}{h_k^n}\sum_{\nu_1=k}\left|\overline{G}_{(\nu)}(\xi)\overline{G}_{(\nu)}(\eta)\right|\le\frac{1}{\omega_n}. \tag{8}$$

Proof. Plugging $|\xi|=1$ and $x=\varepsilon\xi$ into (5), for $\varepsilon\in(0,1)$ we get

$$\frac{1}{(n-2)\omega_n}\,(1-\varepsilon)^{2-n}=\sum_{k=0}^{\infty}\frac{\varepsilon^k}{2k+n-2}\sum_{\nu_1=k}\left(\overline{G}_{(\nu)}(\xi)\right)^2.$$

Hence, keeping in mind that

$$(1-\varepsilon)^{2-n}=\sum_{k=0}^{\infty}\binom{k+n-3}{n-3}\varepsilon^k$$

we obtain

$$\sum_{\nu_1=k}\left(\overline{G}_{(\nu)}(\xi)\right)^2=\frac{1}{\omega_n}\frac{2k+n-3}{n-2}\binom{k+n-3}{n-3}=\frac{h_k^n}{\omega_n}.$$

To get the desired estimate (8) it is sufficient to use the inequality

$$2\left|\overline{G}_{(\nu)}(\xi)\overline{G}_{(\nu)}(\eta)\right|\le\overline{G}_{(\nu)}^{\,2}(\xi)+\overline{G}_{(\nu)}^{\,2}(\eta).$$

This ends the proof. □

Theorem 4. *Let $u(x)$ be a harmonic function in S_n and a continuous one in $\overline{S}_n$. Then for $x\in S_n$ the expansion*

$$u(x)=\sum_{\nu}u_\nu\,\overline{G}_{(\nu)}(x), \tag{9}$$

where u_ν are Fourier coefficients of $u(x)$ by the system $\{\overline{G}_{(\nu)}\}$, is true and we can differentiate it any time under the summation sign.

Proof. It is not hard to verify that for $x\ne\xi$ and $\Lambda=x_1\partial/\partial x_1+\cdots+x_n\partial/\partial x_n$ we have

$$\frac{1}{\omega_n}\,\frac{1-|x|^2}{|x-\xi|^n}=\Lambda_xE(x,\xi)-\Lambda_\xi E(x,\xi),$$

and hence for $|x|<|\xi|=1$, according to Lemma 3, we obtain

$$\frac{1}{\omega_n}\,\frac{1-|x|^2}{|x-\xi|^n}=\sum_{\nu}\overline{G}_{(\nu)}(x)\overline{G}_{(\nu)}(\xi). \tag{10}$$

Differentiation and passage to the limit under the summation sign is valid because, by virtue of Lemma 4, the series in (10) is uniformly convergent with respect to both $\xi\in\overline{S}_n$ and $|x|\le\alpha<1$. Plugging (10) into the Poisson formula for the Dirichlet problem in S_n we arrive at (9). Differentiation under the summation sign in (9) is valid also by virtue of Lemma 4. □

Let $P(x)$ be obtained from $\mathcal{P}$; then denote by $|P|_D$ its norm in $\mathcal{P}$.

Corollary 2. *Let $u(x)$ be a harmonic function in S_n and a continuous one in $\overline{S}_n$. Then*

$$G_{(\nu)}(D)u(x)_{|x=0}=g_\nu\int_{|\xi|=1}G_{(\nu)}(\xi)u(\xi)\,d\xi, \tag{11}$$

where $g_\nu=|G_{(\nu)}|_D^2/\|G_{(\nu)}\|^2$.

Proof. Consider the equality (9). Applying to it the operator $G_{(\mu)}(D)$ and setting $x = 0$ we obtain

$$G_{(\mu)}(D)u(x)_{|x=0} = \sum_{\nu_1=\mu_1} \frac{\langle G_{(\mu)}, G_{(\nu)} \rangle}{\|G_{(\nu)}\|^2} \int_{|\xi|=1} G_{(\nu)}(\xi)u(\xi)\, d\xi.$$

If we now use Theorem 3, then we immediately get (11). The proof is complete. □

Example. Let $u(x)$ be as described in Corollary 2. Then

$$\frac{1}{\omega_n} \int_{|\xi|=1} \xi_i u(\xi)\, d\xi = \frac{1}{n} u_{x_i}(0), \quad \frac{1}{\omega_n} \int_{|\xi|=1} \xi_i \xi_j u(\xi)\, d\xi = \frac{1}{n(n+2)} u_{x_i x_j}(0),$$
$$\frac{1}{\omega_n} \int_{|\xi|=1} \xi_i^2 u(\xi)\, d\xi = \frac{1}{n(n+2)} u_{x_i x_i}(0) + \frac{1}{n} u(0).$$

These equalities can be easily obtained from (11) if we take $G_{(\nu)}(x) = x_i$ (when $\nu = e_1 + \cdots + e_{n-i+1}$, where $e_i = (\delta_{1,i}, \ldots, \delta_{n,i})$), $G_{(\nu)}(x) = x_i x_j$ (when $\nu = 2e_1 + \cdots + 2e_{n-j+1} + e_{n-j+2} + \cdots + e_{n-i+1}$ if $i > j$), and $G_{(\nu)}(x) = x_n^2/2 - |\tilde{x}|^2/(2n-2)$ (when $\nu = 2e_1$) respectively.

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