A CLASS OF $M$-DILATION SCALING FUNCTIONS
WITH REGULARITY GROWING PROPORTIONALLY
TO FILTER SUPPORT WIDTH

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Abstract. In this paper, a class of $M$-dilation scaling functions with regularity growing proportionally to filter support width is constructed. This answers a question proposed by Daubechies on p.338 of her book Ten Lectures on Wavelets (1992).

1. Introduction

Let $M \geq 2$ be a fixed integer. A multiresolution analysis for dilation $M$ consists of a sequence of closed subspaces $V_j$ of $L^2(\mathbb{R})$ that satisfy the following conditions (see [C], [D], [M]):

i) $V_j \subset V_{j+1}$, $\forall j \in \mathbb{Z}$;

ii) $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})$;

iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;

iv) $f \in V_j$ if and only if $f(2^{-j} \cdot) \in V_0$;

v) there exists a function $\phi$ in $V_0$ such that $\{\phi(\cdot - n); n \in \mathbb{Z}\}$ is an orthonormal basis of $V_0$.

The function $\phi$ is called an $M$-dilation scaling function. It is easy to see that $\phi$ satisfies the refinement equation

$$\phi(x) = \sum_{n \in \mathbb{Z}} c_n \phi(Mx - n),$$

where the sequence $\{c_n\}$ satisfies

$$\sum_{n \in \mathbb{Z}} c_n = M.$$
In this paper we shall only deal with compactly supported $M$-dilation scaling functions. In this case the sequence $\{c_n\}$ must have finite length. The function

$$H(\xi) = \frac{1}{M} \sum_{n \in \mathbb{Z}} c_n e^{in\xi}$$

is called a symbol corresponding to the refinement equation (1).

The filter support width $W(\phi)$ of an $M$-dilation scaling function $\phi$ is defined as the difference of the largest and the smallest indices of the nonzero $c_n$. The regularity $R(\phi)$ of $\phi$ is defined as the supremum of $\alpha$ such that $\phi \in C^\alpha$, where $C^\alpha$ denotes the Hölder class of index $\alpha$.

In her book [D, p.338], Daubechies remarks that:

At present, I know of no explicit scheme that provides an infinite family of $m_0$ (i.e., symbol $H$), for dilation $3$ (i.e., $M = 3$), with regularity growing proportionally to the filter support width.

To our knowledge, this question is still open. The purpose of this paper is to construct a class of $M$-dilation scaling functions $\phi_N$ for which there exists a constant $\lambda_M$ independent of $N$ such that

$$R(\phi_N) \geq \lambda_M W(\phi_N),$$

where $M \geq 3$. On the other hand it is already known that (see [DL])

$$W(\phi_N) \geq R(\phi_N).$$

These facts give an affirmative answer to Daubechies’ question.

The regularity of $\phi$ has been studied widely; see for instance [CL], [BDS], [D], [HW1], [HW2], [So], [S] and [WL]. In general, to study the regularity of $\phi$ we need to consider the symbol (2) first. By the Fourier transform, we see that all the symbols $H$ satisfy

$$\sum_{l=0}^{M-1} |H(\xi + \frac{2l\pi}{M})|^2 = 1.$$  \hspace{1cm} (4)

The solutions $H$ of the equation (4) are determined by (see [BDS], [H], [HSZ])

$$|H(\xi)|^2 = \left(\frac{\sin^2(M\xi/2)}{M^2 \sin^2 \xi/2}\right)^N \sum_{s=0}^{N-1} M_a(s) \sin^{2s} \frac{\xi}{2} + (\sin \frac{M\xi}{2})^{2N} R(\xi),$$ \hspace{1cm} (5)

where

$$M_a(s) = \sum_{s_1 + \ldots + s_{M-1} = s} \prod_{j=1}^{M-1} \left(\frac{N - 1 + s_j}{s_j}\right) \frac{1}{\sin^{2s_j} j\pi/M}$$

and $R$ is a real-valued trigonometric polynomial such that $\sum_{l=0}^{M-1} R(\xi + 2l\pi/M) = 0$ and the right hand side of (5) is nonnegative.

By the Riesz Lemma (see [D, p.172]), such symbol $H$ exists. Let $\phi_N$ be a solution of (5) with $R = 0$, and let $\phi_N$ be the solution of (1) corresponding to the symbol $\phi_N$. In [BDS], Bi, Dai and Sun prove the following estimates on the regularity of $\phi_N$:  

$$|R(\phi_N) - \frac{\ln N}{4\ln M}| \leq C$$
when $M$ is odd, and

$$|R(N\phi) - 4N \ln \left( \sin \frac{M\pi}{2M+2} \right)^{-1} + \ln N | \leq C$$

when $M$ is even. A more precise estimate of $R(N\phi)$ can be found in [S]. For the special cases $M = 3, 4, 5$, similar results are obtained by Soardi ([So]) and Heller and Wells ([HW2]) independently. This result shows that for these special $N\phi$ the regularity does not grow proportionally to the filter support width when $M$ is odd.

To construct $M$-dilation scaling functions with regularity growing proportionally to the filter support width we use the symbol $H_N$ determined by

$$|H_N(\xi)|^2 = \sum_{k_0 + \cdots + k_{M-1} = MN-M+1} \alpha_N(k_0, \ldots, k_{M-1}) \frac{(MN - M + 1)!}{k_0! \cdots k_{M-1}!} \prod_{l=0}^{M-1} \frac{\sin M\xi/2}{M \sin(\xi/2 + l\pi/M)}^{2k_l},$$

where $N \geq 1$, and $\alpha_N(k_0, \ldots, k_{M-1})$ is defined by

$$\alpha_N(k_0, \ldots, k_{M-1}) = \begin{cases} 0, & \text{if } k_0 \leq N - 1, \\ 1/\pi^{2l}, & \text{if } k_0 \geq N, \end{cases}$$

where $E = \{j : k_j \geq N\}$ and $\#(E)$ is the cardinality of $E$. Let $\phi_N$ be the solution of (1) corresponding to a symbol $H_N$. Then we have the following

**Theorem.** Let $M \geq 3$ and $N \geq 2$ be any natural numbers. Then $\phi_N$ is an $M$-dilation scaling function and there exists a constant $C$ independent of $N$ such that

$$\frac{1}{2} - \frac{M-1}{2 \ln M} \left( \frac{1}{M-1} \right) N - \frac{\ln N}{4 \ln M} - C \leq R(\phi_N) \leq \frac{1}{2} - \frac{M-1}{2 \ln M} \left( \frac{1}{M-1} \right) N - \frac{\ln N}{4 \ln M} + C.$$

**Remark 1.** Observe that $W(\phi_N) \leq 2(M-1)MN$. Therefore the regularity $R(\phi_N)$ of $\phi_N$ grows proportionally to the filter support width $W(\phi_N)$, i.e., (3) holds.

**Remark 2.** Let $D(\phi) = R(\phi)/W(\phi)$ be the rate of regularity and filter support width of a scaling function $\phi$. Then

$$D(\phi_N) \geq \frac{1}{4M(M-1)} \left( 1 - \frac{(M-1) \ln(1 + \frac{1}{M-1})}{\ln M} \right) - C \frac{\ln N}{N},$$

and

$$D(N\phi) \leq \frac{\ln N}{4NM\ln M} + \frac{C}{N}$$

when $M$ is odd, and

$$D(\phi_N) \leq \frac{\ln(\sin M\pi/(2M+2))^{-1}}{M \ln M} + C \frac{\ln N}{N}$$

when $M$ is even. Therefore we get

$$D(\phi_N)/D(N\phi) \geq \frac{N}{\ln N} \left( \frac{\ln M}{M-1} - \ln(1 + \frac{1}{M-1}) \right) - C$$

when $M$ is odd, and

$$D(\phi_N)/D(N\phi) \geq \frac{\ln M - (M-1) \ln(1 + \frac{1}{M-1})}{4(M-1) \ln(\sin M\pi/(2M+2))^{-1}} - C \frac{\ln N}{N}$$

when $M$ is even.
when $M$ is even. This shows that $D(\phi_N)$ of the $M$-dilation scaling function $\phi_N$ is larger than the one of $N\phi$ even when $M$ is an even integer larger than 4.

2. Proof of the Theorem

To prove the Theorem, we estimate $H_N(\xi)$ first. Let
\[ h(\xi) = \frac{\sin^2 M\xi/2}{M^2 \sin^2 \xi/2} \]
and
\[ B_N(\xi) = (h(\xi))^{-N} |H_N(\xi)|^2. \]

Then for all real valued $\xi$ we have
\[ B_N(-\xi) = \sum_{k_0 + \cdots + k_{M-1} = MN-M+1} \alpha_N(k_0, k_{M-1}, \ldots, k_1) \frac{(MN-M+1)!}{k_0! k_1! \cdots k_{M-1}!} \times (h(\xi))^{k_0-N} \prod_{l=1}^{M-1} (h(\xi + \frac{2l\pi}{M}))^{k_l} \]
\[ = B_N(\xi) \]
and
\[ B_N(\xi) \geq 0. \]

Therefore by the Riesz Lemma ([D, p.172]) we obtain the existence of $H_N(\xi)$ with
\[ H_N(\xi) = \left( \frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})} \right)^N \tilde{H}_N(\xi) \]
and
\[ |\tilde{H}_N(\xi)|^2 = B_N(\xi). \]

From the definition of $\alpha_N(k_0, \ldots, k_{M-1})$ and from
\[ \sum_{l=0}^{M-1} h(\xi + \frac{2l\pi}{M}) = 1, \]
we get
\[ \alpha_N(k_0, k_1, \ldots, k_{M-1}) + \alpha_N(k_1, k_2, \ldots, k_{M-1}, k_0) + \cdots + \alpha_N(k_{M-1}, k_0, \ldots, k_{M-2}) = 1 \]
and
\[ \sum_{l=0}^{M-1} |H_N(\xi + \frac{2l\pi}{M})|^2 \]
\[ = \sum_{k_0 + \cdots + k_{M-1} = MN-M+1} \left( \alpha_N(k_0, k_1, \ldots, k_{M-1}) + \alpha_N(k_1, k_2, \ldots, k_{M-1}, k_0) + \cdots + \alpha_N(k_{M-1}, k_0, \ldots, k_{M-2}) \right) \times \frac{(MN-M+1)!}{k_0! k_1! \cdots k_{M-1}!} \prod_{l=0}^{M-1} (h(\xi + \frac{2l\pi}{M}))^{k_l} \]
\[ = ( \sum_{l=0}^{M-1} h(\xi + \frac{2l\pi}{M}) )^{MN-M+1} \]
\[ = 1. \]
Therefore (4) holds for $H_N(\xi)$. Recall that $H_N(\xi) \neq 0$ when $|\xi| \leq \pi/M$. Hence the solution $\phi_N$ of (1) corresponding to the symbol $H_N(\xi)$ is an $M$-dilation scaling function by an elementary argument ([D, p.182, Theorem 6.3.1] with $K = [-\pi, \pi]$).

To estimate the regularity of $\phi_N$, we need some estimates on $B_N(\xi)$. From the Stirling formula, which says that $n!$ is equivalent to $n^n e^{-n} \sqrt{n}$, from $1/(M - 1) \leq \alpha_N(k_0, \cdots, k_{M-1}) \leq 1$ when $k_0 \geq N$ and from $h(2\pi/(M - 1)) = 1/M^2$, we get

$$B_N(\xi) \leq \sum_{k_0 + \cdots + k_{M-1} = MN - M + 1, k_0 \geq N} (MN - M + 1)! \over k_0! \cdots k_{M-1}! \times (h(\xi))^{k_0} \prod_{l=1}^{M-1} (h(\xi + 2l\pi/M))^{k_l} \leq \frac{(MN - M + 1)!}{N!(N-1)(M-1)!} \leq CM^N_1 + \frac{1}{M - 1} (M-1)^N N^{-1/2}$$

and

$$B_N(2\pi/(M - 1)) \geq \frac{1}{M - 1} \sum_{k_1 + \cdots + k_{M-1} = (M - 1)(N-1)} (MN - M + 1)! \over N!k_1! \cdots k_{M-1}! \times \prod_{l=1}^{M-1} (h(2\pi/(M - 1) + 2l\pi/M))^{k_l} \geq \frac{1}{M - 1} \times \frac{(MN - M + 1)!}{N!(N-1)(M-1)!} (1 - \frac{1}{M^2})^{(N-1)(M-1)} \geq CM^N_1 + \frac{1}{M} (M-1)^N N^{-1/2}.$$

Therefore we get

$$R(\phi_N) \geq \left(\frac{1}{2} - \frac{(M - 1) \ln(1 + \frac{1}{M})}{2 \ln M}\right)N - \frac{\ln N}{4 \ln M} - C$$

by an argument as in [D, p.217]. Observe that $2M\pi/(M - 1) = 2\pi/(M - 1) + 2\pi$.

By an argument similar to that on p.220 of [D] we obtain

$$R(\phi_N) \leq \left(\frac{1}{2} - \frac{(M - 1) \ln(1 + 1/M)}{2 \ln M}\right)N - \frac{\ln N}{4 \ln M} - C.$$

This completes the proof of the Theorem.

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