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TAUTNESS AND COMPLETE HYPERBOLICITY OF DOMAINS IN $\mathbb{C}^n$

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Abstract. We prove that the existence of a local peak holomorphic function at each boundary point of an unbounded domain and at infinity implies the complete hyperbolicity of this domain, and we give a link between local tautness and global tautness of a domain. We end the note with some examples of taut and complete hyperbolic domains arising from the study of domains with noncompact automorphisms group.

Introduction

Many authors were interested in the classification of domains satisfying topological conditions. For instance B.Wong [Wo], J.P.Rosay [Ro], E.Bedford-S.Pinchuk [B-P1], [B-P2], R.Greene-S.G.Krantz [G-K] and F.Berteloot [Be] characterized, in terms of models, some domains with noncompact automorphisms group; their main assumption is that a boundary point is an accumulating point for an orbit of the automorphism group of the domain. The normality of certain holomorphic maps is a striking argument in many results in this field.

The convergence of holomorphic maps between two domains is closely related to the form of the target domain. For instance it is known that in one complex variable any family of holomorphic maps missing two common values is normal. Few similar results of this type are available in several complex variables. However the notion of hyperbolicity of a domain (see [Ko]) seems relevant to getting information on the behavior of holomorphic maps (see the next section for precise definitions).

In this paper we give sufficient conditions on a domain, in terms of hyperbolicity, implying the normality of holomorphic maps with values in this domain. Since unbounded domains are of great interest for understanding the local geometry of a domain near a boundary point, we will deal with such domains.

In the first section we give the definitions of the different notions, and we present our main results. In the second part we prove these results. Finally, in the last section we deduce some examples illustrating our results. The boundary near an accumulating point plays an important part in the characterization of domains with noncompact automorphisms group. We will focus on such domains and on polynomial domains obtained after a scaling process, hoping that our results will be useful to people working on that subject.

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1. Definitions and main results

1.1. **Definitions.** The notations we use are essentially the same as in [Ro].

Definition 1.1. Let p, q be two points in a domain Ω in $\mathbb{C}^n$ and X a vector in $\mathbb{C}^n$.

(a) The **Kobayashi infinitesimal pseudometric** $F_\Omega(p, X)$ is defined by

$$F_\Omega(p, X) = \inf\{\alpha > 0 | \exists g : \Delta \overset{hol.}{\to} \Omega, g(0) = p, g'(0) = X/\alpha\}.$$

(b) The **Kobayashi pseudodistance** $d^K_\Omega(p, q)$ is defined by

$$d^K_\Omega(p, q) = \inf \int_0^1 F_\Omega(\gamma(t), \gamma'(t))dt,$$

where the infimum is taken over all differentiable curves joining p to q.

For instance, in the unit disc Δ in $\mathbb{C}$ the form F_Δ agrees with the differential form of the Poincaré metric:

$$F_\Delta(p, X) = \frac{|X|}{1 - |p|^2},$$

and the Kobayashi distance d^K_Δ is given by

$$d^K_\Delta(0, p) = \frac{1}{2} \ln \frac{1 + |p|}{1 - |p|}.$$

Definition 1.2. (a) A domain Ω is **hyperbolic** if d^K_Ω is a distance on Ω.

(b) A hyperbolic domain Ω is **complete hyperbolic** if it is complete for the distance d^K_Ω.

H.L.Royden proved in [Ro] (theorem 2, p.133) that a domain Ω is hyperbolic if for every point p in Ω there exist a neighborhood U of p and a positive constant c such that $F_\Omega(y, X) \geq c\, ||X||$ for all $y \in U$. In that case the family of holomorphic maps from Δ to Ω is equicontinuous for the distance d^K_Ω. He also noticed (proposition 7, p.136) that a hyperbolic domain Ω is complete hyperbolic if for each point z in Ω and for each positive real number r the Kobayashi ball $\{y \in \Omega : d^K_\Omega(z, y) \leq r\}$ is compact in Ω.

Definition 1.3. (a) A sequence $(f_\nu)_\nu$ of holomorphic maps from a domain D to a domain G is **compactly divergent** if for all compact subsets K of D and K' of G there exists a positive integer ν_0 such that :

$$\nu \geq \nu_0 \Rightarrow f_\nu(K) \cap K' = \emptyset.$$

(b) A family $\mathcal{A}$ of maps from D to G is **normal** if each sequence of $\mathcal{A}$ admits a subsequence that is either uniformly convergent on compact subsets of D or compactly divergent.

(c) A domain Ω is **taut** if the family $\mathcal{H}(\Delta, \Omega)$ of holomorphic maps from Δ to Ω (called analytic discs) is normal.

(d) A domain Ω is **locally taut** at a point p in $\partial\Omega$ if there exists a neighborhood U of p such that $\Omega \cap U$ is taut.

The relation between hyperbolicity, tautness and complete hyperbolicity of a domain is given by the following proposition ([Ro], proposition 5, p.135, and the corollary on p.136):

Proposition. *Ω complete hyperbolic $\Rightarrow$ Ω taut $\Rightarrow$ Ω hyperbolic.*

We recall that a neighborhood of infinity in an unbounded domain Ω is a set containing the complement of a closed ball in Ω. If φ is a function defined on Ω and c a complex number, we set $\varphi(\infty) = c$ if $\lim_{|z|\to\infty} \varphi(z) = c$.

Definition 1.4. (a) A function φ (resp. h) is called a **local peak plurisubharmonic** (resp. **holomorphic**) **function** at a point p in $\partial\Omega \cup \{\infty\}$ if there exists a neighborhood U of p such that φ (resp. h) is plurisubharmonic (resp. holomorphic) on $\Omega \cap U$, continuous up to $\overline{\Omega} \cap U$ and satisfies

$$\begin{cases} \varphi(p) = 0 & (\text{resp. } \ h(p) = 1), \\ \varphi(z) < 0 & (\text{ resp. } \ |h(z)| < 1) \quad \text{ for all } \ z \in \overline{\Omega} \cap U. \end{cases}$$

(b) A function Ψ is called a **local antipeak plurisubharmonic** function at a point p in $\partial\Omega \cup \{\infty\}$ if there is a neighborhood U of p such that Ψ is plurisubharmonic on $\Omega \cap U$, continuous up to $\overline{\Omega} \cap U$ and satisfies

$$\begin{cases} \Psi(p) & = & -\infty, \\ \Psi(z) & > & -\infty \quad \text{ for all } z \in (\overline{\Omega} \cap U)\backslash\{p\}. \end{cases}$$

Remark 1.5. The existence of an antipeak plurisubharmonic function at a finite point p can always be ensured by setting $\Psi(z) = \ln|z - p|$.

1.2. **Main results.** Our first result gives a relation between the existence of a peak holomorphic function at each boundary point of a domain and the complete hyperbolicity of this domain:

Theorem 1. *Let Ω be a domain in $\mathbb{C}^n$. Assume that there is a local peak holomorphic function at each point p in $\partial\Omega \cup \{\infty\}$. Then Ω is a complete hyperbolic domain.*

This is a generalization of well-known results on bounded domains. For instance, a bounded domain admitting a global peak holomorphic function at each boundary point is complete hyperbolic ([B-F], theorem 3.5, p.563). If this is a consequence of the definition of the Carathéodory metric since it deals with global functions, the situation is different in theorem 1 because of the local assumptions.

The second result gives a link between local tautness and global tautness of a domain. However, since we deal with unbounded domains, we need a precise control of analytic discs at infinity.

Proposition 2. *Let Ω be a domain in $\mathbb{C}^n$. Assume that Ω is locally taut at each point in $\partial\Omega$ and that there are local peak and antipeak plurisubharmonic functions at infinity. Then Ω is a taut domain.*

This theorem can be viewed as a general version of a result of F.Berteloot [Be] obtained for domains with noncompact automorphisms group.

As a consequence of proposition 2 we obtain the following corollary:

Corollary 3. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with a Lipschitz boundary. Assume that there are local peak and antipeak plurisubharmonic functions at infinity. Then Ω is a taut domain.*

We recall that a domain has a Lipschitz boundary if its boundary is locally defined by a Lipschitz function. As a particular case of domains satisfying the assumptions of corollary 3 one can consider domains with a Lipschitz boundary, admitting a local peak holomorphic function at infinity. This corollary generalizes a result of N.Kerzman ([Ke], theorem on p.676) stating that a smooth bounded pseudoconvex domain of class $\mathcal{C}^1$ is a taut domain. One can note that pseudoconvexity is a necessary condition, since every taut domain is pseudoconvex ([Wu], theorem F, p.211).

2. Proof of the results

2.1. **Proof of theorem 1.** To prove theorem 1 we have to show that Ω is hyperbolic and that the Kobayashi balls are compact in Ω, in view of [Ro] (proposition 7, p.136). Since a bounded domain is hyperbolic, the hyperbolicity of Ω is based here on the existence of a local peak function at infinity. This is obtained by a localization of analytic discs. Such a localization at infinity or at a boundary point needs weaker assumptions than the existence of a holomorphic peak function; we prove this in lemma 1. The hyperbolicity of Ω is proved in lemma 2. In lemma 3 we study the behavior of the Kobayashi pseudodistance near the boundary of Ω and at infinity; this study uses deeply the existence of a local peak holomorphic function. We end this subsection with the proof of theorem 1.

Lemma 2.1.1. *Let p be a point in $\partial\Omega\cup\{\infty\}$. Assume that there are local peak and antipeak plurisubharmonic functions φ and Ψ at p, both defined on a neighborhood V_p of p. Then for every neighborhood U of p there exists a neighborhood U' of p such that every analytic disc in Ω satisfies*

$$f(0)\in U' \Rightarrow f(\Delta_{1/2})\subset U,$$

where $\Delta_{1/2}=\{\zeta\in\Delta : |\zeta|<1/2\}$.

Remark 1. Lemma 2.1.1 proves that any family $(f_\nu)_\nu$ of analytic discs such that $\lim_{\nu\to\infty} f_\nu(0)=p$ converges to p uniformly on the disc $\Delta_{1/2}$.

Proof. Since φ is a local peak plurisubharmonic function at p, there exist two neighborhoods $U,\ V$ of p ($\overline{U}\subset V\subset V_p$) and two positive constants $c,\ c'$ ($c>c'$) such that:

$$\begin{cases} \inf_{z\in\overline{\Omega}\cap bU} \quad \varphi(z) &= -c', \\ \sup_{z\in\overline{\Omega}\cap bV} \quad \varphi(z) &= -c. \end{cases}$$

Then the function $\tilde{\varphi}$ defined on $\overline{\Omega}$ by

$$\begin{cases} \tilde{\varphi}(z) &= \varphi(z) & \text{if } z\in\overline{\Omega}\cap U, \\ \tilde{\varphi}(z) &= \sup(\varphi(z),-(c+c')/2) & \text{if } z\in\overline{\Omega}\cap(V\backslash\overline{U}), \\ \tilde{\varphi}(z) &= -(c+c')/2 & \text{if } z\in\overline{\Omega}\backslash V, \end{cases}$$

is a global negative peak plurisubharmonic funtion at p.

Let f be an analytic disc in Ω. Since the function $\tilde{\varphi}\circ f$ is subharmonic, the mean value inequality implies that for every negative α such that $(\tilde{\varphi}\circ f)(0)>\alpha$, the measure of the set $E_\alpha=\{\theta\in[0,2\pi]/(\tilde{\varphi}\circ f)(e^{i\theta})\geq 2\alpha\}$, denoted by $mes(E_\alpha)$, satisfies

$$mes(E_\alpha)\geq\pi. \tag{1}$$

Consider now a sufficient small positive constant ε such that:

$$\begin{cases} \inf_{\overline{\Omega}\cap bU}\ (\varphi+\varepsilon\Psi) &= -c_1<0, \\ \sup_{\overline{\Omega}\cap bV}\ (\varphi+\varepsilon\Psi) &= -c_2<-c_1. \end{cases}$$

The function ρ defined on $\overline{\Omega}$ by:

$$\begin{cases} \rho(z) &= (\varphi+\varepsilon\Psi)(z) & \text{if } z\in\overline{\Omega}\cap U, \\ \rho(z) &= \sup((\varphi+\varepsilon\Psi)(z),-(c_1+c_2)/2) & \text{if } z\in\overline{\Omega}\cap(V\backslash\overline{U}), \\ \rho(z) &= -(c_1+c_2)/2 & \text{if } z\in\overline{\Omega}\backslash V, \end{cases}$$

is a continuous negative plurisubharmonic function on Ω and satisfies $\rho^{-1}(-\infty)=\{p\}$. Consequently, using the Poisson integral, for any point ζ in $\Delta_{1/2}$ we get

$$(\rho\circ f)(\zeta)\leq\frac{3}{10\pi}\int_0^{2\pi}(\rho\circ f)(e^{i\theta})d\theta. \tag{2}$$

Since $\tilde{\varphi}$ is a peak function at p and ρ satisfies $\rho(p)=-\infty$, there exists for each positive constant L a negative constant α such that for any point z in $\overline{\Omega}$ the inequality $\tilde{\varphi}(z)\geq 2\alpha$ implies $\rho(z)<-L$. Consequently, using inequalities (1), (2) and the fact that ρ is a negative function, we obtain, for every analytic disc f in Ω and for every point ζ in $\Delta_{1/2}$,

$$\tilde{\varphi}(f(0))>\alpha\Rightarrow(\rho\circ f)(\zeta)\leq-\frac{3}{10}L. \tag{3}$$

Since $\rho^{-1}(-\infty)=\{p\}$, the family $(U_n=\{z\in\overline{\Omega}:\rho(z)<-(3/10)n\})_n$ is a neighborhood basis of p in $\overline{\Omega}$. Consider for each positive integer n there is a negative constant α_n such that

$$\tilde{\varphi}(z)\geq 2\alpha_n\Rightarrow\rho(z)<-n.$$

Let U'_n be the neighborhood of p in $\overline{\Omega}$ defined by $U'_n=\{z\in\overline{\Omega}:\tilde{\varphi}(z)>\alpha_n\}$. Using equation (3), we obtain for every n

$$f(0)\in U'_n\Rightarrow f(\Delta_{1/2})\subset U_n.$$

This proves lemma 2.1.1. □

A local assumption on Ω at infinity implies the tautness of the domain:

Lemma 2.1.2. *Let Ω be a domain in $\mathbb{C}^n$. Assume that there are local peak and antipeak plurisubharmonic functions at infinity. Then Ω is a hyperbolic domain.*

Proof. Assume for a contradiction that Ω is not hyperbolic. Then there exist a point z^0 in Ω, a sequence $(y_\nu)_\nu$ of points in Ω converging to z^0, and a sequence $(V_\nu)_\nu$ of unitary vectors in $\mathbb{C}^n$ such that $F_\Omega(y_\nu,V_\nu)\leq 1/\nu$. Consequently there exists a sequence $(f_\nu)_\nu$ of analytic discs centered at y_ν such that $|f'_\nu(0)|\geq\nu$. By Montel's theorem there is a sequence $(p_\nu)_\nu$ of points in Δ converging to 0 such that $\lim_{\nu\to\infty}|f_\nu(p_\nu)|=\infty$. Composing every f_ν with a Moebius transform, we get a family $(\tilde{f}_\nu)_\nu$ of analytic discs such that $\tilde{f}_\nu(0)=f_\nu(p_\nu)$ and $\tilde{f}_\nu(p_\nu)=y_\nu$. This contradicts the localization of lemma 2.1.1. □

We obtain now the behavior of the Kobayashi pseudodistance d^K_Ω near each boundary point and at infinity:

Lemma 2.1.3. *Let Ω be a domain in $\mathbb{C}^n$ satisfying the assumptions of theorem 1. Let p be a point in $\partial\Omega \cup \{\infty\}$. Then for every neighborhood U of p in $\mathbb{C}^n$ we get*

$$\lim_{z \to p} d_\Omega^K(z, z') = \infty$$

for any point z' in $\overline{\Omega} \cap \partial U$.

Proof. Let h be a local peak holomorphic function at p defined on a neighborhood V_p of p. Since the function $\Psi(z) = \ln|1 - h(z)|$ is a local antipeak plurisubharmonic function at p, lemma 2.1.1 implies that for each neighborhood U of p there is a neighborhood U' of p such that for every analytic disc f in Ω

$$f(0) \in U' \Rightarrow f(\Delta_{1/2}) \subset U.$$

In particular, for every z in U' and X in $\mathbb{C}^n$ we get

$$F_\Omega(z, X) \geq (1/2) F_U(z, X).$$

Let z' be a point in $\Omega \cap \partial U$. If γ is a differentiable curve in Ω from z' to z, there exists $t_0 \in [0, 1[$ such that the point $\gamma(t_0)$ is in $\Omega \cap \partial U'$ and the set $\gamma(]t_0, 1])$ is contained in $\Omega \cap U'$. Then we obtain

$$\int_0^1 F_\Omega(\gamma(t), \gamma'(t))dt \geq \frac{1}{2} \int_{t_0}^1 F_U(\gamma(t), \gamma'(t))dt.$$

Since the map h is holomorphic from $\Omega \cap V_p$ to Δ and $\sup_{z \in \Omega \cap \partial U} |h(z)| = C < 1$, we get

$$\int_0^1 F_\Omega(\gamma(t), \gamma'(t))dt \geq (1/2) \int_{t_0}^1 F_\Delta(h(\gamma(t)), dh(\gamma(t)).\gamma'(t))dt \geq (1/2) \ln \frac{1 - C}{1 - |h(z)|}.$$

The limit when z converges to p of the last quantity is infinity; this is also satisfied for the Kobayashi distance d_Ω^K. □

We can now prove theorem 1:

Proof of theorem 1. Lemma 2.1.2 shows that the domain Ω is hyperbolic, and lemma 2.1.3 proves that the Kobayashi ball $\{y \in \Omega : d_\Omega^K(z, y) \leq r\}$ is compact in Ω for every point z in Ω and every positive constant r. Then Ω is complete hyperbolic. □

2.2. **Proof of proposition 2.** We begin with a uniform behavior of automorphisms of the unit disk. For any point ζ in Δ and any real number θ in $[0, 2\pi[$ we call $g_{\zeta,\theta}$ an automorphism of Δ, where

$$g_{\zeta,\theta}(\lambda) = \frac{\zeta - e^{i\theta}\ \lambda}{1 - e^{i\theta}\ \lambda\overline{\zeta}}.$$

Lemma 2.2.1. *For any relatively compact set K in Δ there is a positive real constant r_K such that every automorphism g of Δ satisfies*

$$g(0) \in K \Rightarrow \Delta(g(0), r_K) \subset g(\Delta_{1/2}),$$

where $\Delta(g(0), r_K) = \{\lambda \in \Delta : |\lambda - g(0)| < r_K\}$.

Proof. Let ζ be a point in K, θ a real number in $[0, 2\pi[$ and $g_{\zeta,\theta}$ the associated automorphism of Δ. It is sufficient to prove that $\inf_{|\lambda|=1/2} |g_{\zeta,\theta}(\lambda) - g_{\zeta,\theta}(0)|$ is uniformly bounded on K.

Since

$$|g_{\zeta,\theta}(\lambda) - g_{\zeta,\theta}(0)| \geq (1/2)\ |\lambda|\ (1 - |\zeta|^2),$$

one can set $r_K = (1/4) \min_{\zeta \in K}(1 - |\zeta|^2)$. □

Proof of proposition 2. First case: *there exist a point ζ in Δ and a subsequence $(f_{\nu_k})_k$ of $(f_\nu)_\nu$ such that* $\lim_{k\to\infty} |f_{\nu_k}(\zeta)| = \infty$.

Let $E = \{\lambda \in \Delta : \lim_{k\to\infty} |f_{\nu_k}(\lambda)| = \infty\}$. Since for any point ζ in E we get

$$\lim_{k\to\infty} |f_{\nu_k} \circ g_{\zeta,\theta}| = \infty$$

uniformly on $\Delta_{1/2}$ by lemma 2.1.1, it follows by lemma 2.2.1 that there exists a positive real number r_ζ such that

$$\lim_{k\to\infty} |f_{\nu_k}(\overline{\Delta(\zeta, r_\zeta)})| = \infty.$$

Then E is open in Δ. Moreover if $(\zeta_n)_n$ is a sequence of points in E converging to a point ζ in Δ, the compactness of the set $\overline{\{\zeta_n, n \geq 0\} \cup \{\zeta\}}$ implies by lemma 2.2.1 that there is a positive constant r such that, for every positive integer n, $\lim_{k\to\infty} |f_{\nu_k}| = \infty$ uniformly on $\Delta(\zeta_n, r)$ and so that $\lim_{k\to\infty} |f_{\nu_k}(\zeta)| = \infty$; the set E is closed in Δ. We finally proved that $E = \Delta$, and so for every point ζ in Δ there is a positive real number r_ζ such that $\lim_{k\to\infty} |f_{\nu_k}| = \infty$ uniformly on $\Delta(\zeta, r_\zeta)$. In particular the sequence $(|f_{\nu_k}|)_k$ diverges to infinity uniformly on compact subsets of Δ, and so the sequence $(f_{\nu_k})_k$ is compactly divergent.

Second case: for every point ζ in Δ the sequence $(f_\nu(\zeta))_\nu$ is relatively compact in $\mathbb{C}^n$.

The reasoning of the first case, proving that E is closed, implies that the sequence $(f_\nu)_\nu$ is locally bounded in Δ. Then Montel's theorem implies that this is a relatively compact family and so there exists a subsequence $(f_{\nu_k})_k$ of $(f_\nu)_\nu$ converging to a holomorphic map f from Δ to $\overline{\Omega}$. The behavior of the subsequence $(f_{\nu_k})_k$ will be deduced from the study of the map f given in the following lemma.

Claim. If there is a point $\zeta \in \Delta$ such that $f(\zeta) \in \partial\Omega$, then the set $f(\Delta)$ is contained in $\partial\Omega$.

Proof of the claim. Let E be the closed set $E = \{\lambda \in \Delta : f(\lambda) \in \partial\Omega\}$. By assumption the set E is nonempty. Let λ be a point in E. Since Ω is locally taut at $f(\lambda)$, there is a neighborhood V of $f(\lambda)$ such that $\Omega \cap V$ is taut. Since f is the continuous uniform limit of $(f_{\nu_k})_k$, there are two neighborhoods U of λ and V' of $f(\lambda)$ ($V' \subset\subset V$) such that the set $f_{\nu_k}(U)$ is contained in $\Omega \cap V'$ for sufficiently large k. Then the tautness of $\Omega \cap V$ implies that $f(U) \subset \partial\Omega$, and so E is an open set. Finally, $E = \Delta$. □

We proved in the first case that the sequence $(f_\nu)_\nu$ admits a compactly divergent subsequence, and in the second case that this admits a subfamily that is either compactly divergent or uniformly convergent on compact subsets of Δ. This proves proposition 2. □

Proof of corollary 3. The local tautness of Ω at each point in $\partial\Omega$ is based on the following result of J.P.Demailly ([De], theorem 0.2, pp.519-520):

If Ω is a pseudoconvex domain with a Lipschitz boundary, then for every point p in $\partial\Omega$ there exist a neighborhood V of p, a plurisubharmonic function Ψ on $\Omega \cap V$ and two positive constants A, B such that, for all z in $\Omega \cap V$,

$$A(\ln \delta(z))^{-1} \leq \Psi(z) \leq B(\ln \delta(z))^{-1},$$

where $\delta(z)$ is the euclidean distance from z to Ω.

Such a construction was obtained previously by N.Kerzman and J.P.Rosay ([Ke-R], lemma 2, p.174) essentially for a smooth pseudoconvex domain of class $\mathcal{C}^1$.

The function Ψ can be extended continuously up to $\overline{\Omega} \cap V$ by setting $\Psi = 0$ on $\partial\Omega \cap V$. If $(f_\nu)_\nu$ is a sequence of analytic discs in $\Omega \cap V$, the maximum principle applied to the family $(\Psi \circ f_\nu)_\nu$ implies that $\mathcal{H}(\Delta, \Omega)$ is a normal family and so $\Omega \cap V$ is taut. □

3. Applications of the results

The scaling process introduced by S.Pinchuk [P] is a very useful tool in the characterization of domains with noncompact automorphisms group. If Ω is such a domain in $\mathbb{C}^n$ and p is an accumulating point of an orbit of the automorphisms group of Ω, then the scaling method applied to a certain neighborhood U of p provides a family $(F_\nu)_\nu$ of biholomorphic maps from $\Omega \cap U$ to domains D^ν. Since the domain $F_\nu^{-1}(D^\nu)$ is contained in Ω, the tautness of Ω is of crucial importance to studying the possible convergence of the family $(F_\nu)_\nu$. The convergence of the domains D^ν is also very important. These domains are obtained after a scaling process applied to $\Omega \cap U$, and so they are defined by a renormalization of a local defining function for the domain Ω at the point p. Thus if the sequence $(D^\nu)_\nu$ is convergent the limit domain will be a polynomial domain; consequently the study of such domains is of great interest.

The tautness of polynomial domains is also crucial in order to use a scaling process in connection with the study of holomorphic maps or CR maps, such as the regularity of such maps (see for instance [C-P-S], [C-S], [C-G-S]). This explains why, as applications of theorem 1, proposition 2 and corollary 3, we study domains with noncompact automorphisms group and polynomial domains. A consequence of our results is that all the domains considered in the examples are hyperbolic. As a corollary of this we get the following information on the structure of the automorphisms group of these domains ([Wu], theorem D′, p.208):

Corollary. *Let Ω be a hyperbolic domain. Then the automorphisms group of Ω is a real Lie group.*

3.1. Domains with noncompact automorphisms group.

Let Ω be a domain in $\mathbb{C}^n$. We assume in this subsection that there are a point p in $\partial\Omega$, a point z^0 in Ω and a sequence $(\varphi_\nu)_\nu$ of automorphisms of Ω such that $\lim_{\nu\to\infty} \varphi_\nu(z^0) = p$. Our main hypothesis will be local at the accumulating point p and will be expressed in terms of the existence of local peak holomorphic or plurisubharmonic functions.

Lemma 3.1.1. *Let Ω be a domain in $\mathbb{C}^n$ with noncompact automorphisms group. Let $p \in \partial\Omega$ be an accumulating point for a sequence $(\varphi_\nu)_\nu$ of automorphisms of Ω. If there is a local peak plurisubharmonic function at p, then the sequence $(\varphi_\nu)_\nu$ converges to p uniformly on compact subsets of Ω.*

This was obtained previously by E.Bedford-S.Pinchuk [B-P1] and F.Berteloot [Be]. However we can view this as a reformulation of lemma 2.1.1, working with automorphisms of Ω instead of analytic discs. In particular this implies that the domain is hyperbolic in a neighborhood of p, and thus that it is hyperbolic, since the Kobayashi distance is invariant under the action of biholomorphic maps. F.Berteloot ([Be]) proved that the local tautness of Ω at p implies the tautness of Ω. This is a local version of proposition 2. As applications of this we obtain some examples of taut domains:

Example 3.1.2. Let Ω be a domain with noncompact automorphisms group in $\mathbb{C}^n$. If Ω is smooth pseudoconvex of finite type in the sense of D'Angelo near an accumulating point, then Ω is taut.

We recall that a domain Ω is of finite type in the sense of D'Angelo at $p \in \partial\Omega$ if the order of contact at p between $\partial\Omega$ and any germ of one-dimensional complex variety is bounded independently of the germ (see [D'Ang], the definition on p.62). Moreover, if Ω is of finite type at p, then it is of finite type at each point of its boundary in a neighborhood of p.

Under these assumptions on Ω the local tautness of Ω at p is given by the existence of a local peak plurisubharmonic function at each point near p in $\partial\Omega$ (see [Cho], theorem 3, p. 318).

As a particular case of example 3.1.2 we have

Example 3.1.3. Let φ be a smooth weighted plurisubharmonic function in $\mathbb{C}^{n-1}$. If the domain $\Omega = \{(z_0, z') \in \mathbb{C}^n : Imz_0 + \varphi(z') < 0\}$ is of finite type in the sense of D'Angelo at 0, then this is a taut domain.

A similar result can be found in [Yu] (theorem 3.13, p. 596).

Proof. The weighted homogeneity of φ means that there exists a multi-index $(m_1, ..., m_{n-1})$ such that for all $z' = (z_1, ..., z_{n-1}) \in \mathbb{C}^{n-1}$ and all real positive numbers t we get:

$$\varphi(t^{1/m_1} z_1, ..., t^{1/m_{n-1}} z_{n-1}) = t\varphi(z').$$

Then the functions φ_j defined on Ω by

$$\varphi_j((z_0, z')) = (j^{-1} z_0, j^{-1/m_1} z_1, ..., j^{-1/m_{n-1}} z_{n-1})$$

are automorphisms of Ω accumulating at 0. □

One can also consider in example 3.1.3 weighted homogeneous polynomial domains obtained as a local representation of smooth pseudoconvex domains of finite type in the sense of D'Angelo.

It is not so easy to study the complete hyperbolicity of the domains considered in examples 3.1.2 and 3.1.3, since we get no concrete information on the Kobayashi balls. For instance, some balls could spread at infinity or even accumulate at a distinct boundary point. Without assumptions on the boundary of Ω near this point, we cannot conclude. However in view of lemma 2.1.3 we can prove the complete hyperbolicity of Ω under stronger hypothesises:

Proposition 3.1.4. *Let p be an accumulating point of a domain Ω with noncompact automorphism group and let U be a neighborhood of p. If there is a local peak holomorphic function at each point in $\partial\Omega \cap U$, then Ω is complete hyperbolic.*

Proposition 3.1.4 can be viewed as a local version of theorem 1.

Proof. We know that Ω is hyperbolic. Let z^0 be a point in Ω and r be a positive constant. The sequence $(\varphi_\nu(z^0))_\nu$ converges to p by lemma 3.1.1. According to lemma 2.1.3 there is a positive integer ν_0 such that the set $\varphi_{\nu_0}(\{d^K_\Omega(z^0, y) \leq r\})$ is contained in $\Omega \cap U$ and does not accumulate at a point in $\partial\Omega \cap U$. Then this is a compact set in $\Omega \cap U$, and so the Kobayashi ball $\{d^K_\Omega(z^0, y) \leq r\}$ is compact in Ω. □

Using proposition 3.1.4, we obtain some examples of complete hyperbolic domains.

Example 3.1.5. Let p be an accumulating point of a domain Ω with noncompact automorphisms group in $\mathbb{C}^n$. If Ω is pseudoconvex of semiregular type near p, then Ω is complete hyperbolic.

Pseudoconvex domains of semiregular type, studied by K.Diederich and G. Herbort [D-H] and by J.Yu [Yu], are a generalization of convex domains of finite type in the sense of D'Angelo. The existence of a local peak holomorphic function at each boundary point near p is given for instance by [D-H].

Example 3.1.6. Let p be an accumulating point of a domain Ω with noncompact automorphism group in $\mathbb{C}^2$. If Ω is pseudoconvex of finite type in the sense of D'Angelo at p, then Ω is complete hyperbolic.

The existence of a local peak holomorphic function at each boundary point is due to [B-F] (theorem 3.1, pp.555-556).

3.2. **Rigid polynomial domains.** These domains appear naturally when one uses a scaling process under some assumptions. A rigid polynomial domain is a domain of the following form:

$$\Omega = \{(z_0, z') \in \mathbb{C}^n : Imz_0 + P(z') < 0\},$$

where P is a polynomial in $\mathbb{C}^{n-1}$.

The automorphism group of such a domain is noncompact since it contains the subgroup $(z_0, z') \mapsto (z_0 + t, z')$ for real parameter t. All these automorphisms accumulate at infinity, and so infinity plays an important part here. So we will obtain results on these domains by assumptions at infinity.

Example 3.2.1. Let $\Omega = \{(z_0, z_1) \in \mathbb{C}^2 : Imz_0 + P(z_1) < 0\}$, where P is a subharmonic polynomial in $\mathbb{C}$ without harmonic terms. Then Ω is complete hyperbolic.

Proof. This is a consequence of theorem 1, since there is a local peak holomorphic function at each boundary point of Ω and at infinity by [B-F] (theorem 3.1, pp.555-556). □

A corresponding result in higher dimensions is given by the following example.

Example 3.2.2. Let $\Omega = \{(z_0, z') \in \mathbb{C}^n : Imz_0 + P(z') < 0\}$, where P is a convex polynomial in $\mathbb{C}^{n-1}$ without pluriharmonic terms. Assume that $P(0) = dP(0) = 0$ and that the set $\{P = 0\}$ contains no nontrivial analytic set. Then Ω is complete hyperbolic.

Proof. Under the assumptions of example 3.2.2 the domain Ω is convex of finite type in the sense of D'Angelo. Then there is a local peak holomorphic function at each point in $\partial\Omega$. Since P is convex without pluriharmonic terms, the function Ψ defined on Ω by $\Psi(z_0, z') = (z_0 + i)(z_0 - i)^{-1}$ is a peak holomorphic function at infinity. Then we can apply theorem 1. □

If P is only assumed to be convex, the existence of a peak holomorphic function at infinity is not known. However we have the following result ([Ga], proposition 2.3, p.45):

Example 3.2.3. Let $\Omega = \{(z_0, z') \in \mathbb{C}^n : Imz_0 + P(z') < 0\}$, where P is a convex polynomial in $\mathbb{C}^{n-1}$. Assume that $P(0) = dP(0) = 0$ and that the set $\{P = 0\}$ contains no nontrivial analytic set. Then Ω is taut.

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