LARGE ORBITS IN ACTIONS OF NILPOTENT GROUPS

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Abstract. If a nontrivial nilpotent group \( N \) acts faithfully and coprimely on a group \( H \), it is shown that some element of \( H \) has a small centralizer in \( N \) and hence lies in a large orbit. Specifically, there exists \( x \in H \) such that \( |C_N(x)| \leq (|N|/p)^{1/p} \), where \( p \) is the smallest prime divisor of \( |N| \).

1. Introduction

A well known result of D. S. Passman [3] asserts that if a \( p \)-group \( P \) acts faithfully via automorphisms on a solvable \( p' \)-group \( H \), then there must always exist a 'large' orbit. Equivalently, there always exists \( x \in H \) such that \( |C_P(x)| \) is 'small', and in most cases, \( x \) can be chosen so that \( C_P(x) = 1 \) and \( x \) lies in a regular \( P \)-orbit. Specifically, Passman’s result is that if \( p = 2 \), then there exists \( x \in H \) such that \( |C_P(x)| \leq |P|^{1/2} \), and if \( p > 2 \), one can find \( x \in H \) with \( |C_P(x)| \leq |P|^{1/3} \). In fact, Passman showed that the only situations where a regular orbit can fail to exist are when \( p = 2 \) and \( |H| \) is divisible by a Fermat or Mersenne prime, or when \( p \) is Mersenne and \( |H| \) is even.

At the end of his paper, Passman stated without proof that in all cases, there exists \( x \in H \) such that \( |C_P(x)| \leq |P|^{1/p} \). The main purpose of this note is to provide a proof of this assertion, with a slightly improved bound.

Theorem A. Let \( P \) be a nontrivial \( p \)-group that acts faithfully on a group \( H \), where \( |H| \) is not divisible by \( p \). Then there exists an element \( x \in H \) such that \( |C_P(x)| \leq |P|^{1/p} \). The main purpose of this note is to provide a proof of this assertion, with a slightly improved bound.

Notice that we are not assuming that \( H \) is solvable. No such hypothesis is necessary because of an amazing elementary lemma of B. Hartley and A. Turull that applies whenever \( A \) acts by automorphisms on \( G \), where \( A \) and \( G \) are finite groups having coprime orders. The Hartley-Turull lemma (see Lemma 2.6.2 of [2]) asserts that it is always possible to replace \( G \) by an abelian group \( G_0 \) such that the actions of \( A \) on the elements of \( G \) and on the elements of \( G_0 \) are permutation isomorphic. In fact, \( G_0 \) can be chosen to have square-free exponent, although we shall not need this additional information.

It is not really necessary in Theorem A that the acting group should be a \( p \)-group; the corresponding result holds whenever an arbitrary finite nilpotent group acts coprimely. Theorem A, of course, is included in the following.
Theorem B. Let $N$ be a nontrivial nilpotent $\pi$-group, where $\pi$ is a set of primes, and assume that $N$ acts faithfully on a $\pi'$-group $H$. Then there exists $x \in H$ such that $|C_N(x)| \leq (|N|/p)^{1/p}$, where $p$ is the smallest member of $\pi$.

Because of the Hartley-Turull lemma mentioned earlier, it is no loss to assume that $H$ is abelian when proving Theorem B, and we will do so. Also, it turns out that only the nonabelian Sylow subgroups of $N$ really matter, and hence Theorem B can be strengthened, as follows.

Theorem C. Let $N$ be nilpotent and suppose that it acts faithfully on $H$, where $(|N|, |H|) = 1$. Let $\pi$ be a set of prime numbers containing all primes for which the Sylow subgroup of $N$ is nonabelian. Then there exists an element $x \in H$ such that $|C_N(x)|$ is a $\pi$-number, and if $\pi$ is nonempty, $x$ can be chosen so that $|C_N(x)| \leq (k/p)^{1/p}$, where $k = |N|_\pi$ is the $\pi$-part of $|N|$ and $p$ is the smallest member of $\pi$.

By a ‘$\pi$-number’ in the statement of Theorem C, we mean, of course, a positive integer all of whose prime divisors lie in $\pi$.

2. Coprimeness consequences and Theorem C

We begin with an easy lemma.

(2.1) Lemma. Suppose that $N$ acts faithfully on $H = A \times B$, where $A$ and $B$ are $N$-invariant. Let $M = C_N(A)$, so that $N/M$ acts on $A$, and let $a \in A$ and $b \in B$. Write $u = |C_{N/M}(a)|$ and $v = |C_M(b)|$. Then $|C_N(ab)|$ divides $uv$.

Proof. Let $U = C_N(a)$, so that $U \supseteq M$ and $|U/M| = u$. Also, let $V = C_N(b)$, so that $|V \cap M| = v$. Then

$$|V \cap U : V \cap M| = |(V \cap U) : (V \cap U) \cap M| = |(V \cap U)M : M|,$$

and this divides $|U/M| = u$. We now have

$$|C_N(ab)| = |V \cap U| = |V \cap U : V \cap M| |V \cap M| = |V \cap U : V \cap M|v,$$

and this divides $uv$, as required.

Assuming Theorem B for the moment, we now prove Theorem C.

Proof of Theorem C. Recall that $N$ is nilpotent and acts faithfully and coprimely on $H$. Also, $\pi$ contains all primes for which a Sylow subgroup of $N$ is nonabelian, and we write $k = |N|_\pi$ to denote the $\pi$-part of $|N|$. Furthermore, by the Hartley-Turull lemma, we can assume that $H$ is abelian.

We associate a real number $f(r)$ to an arbitrary $\pi$-number $r$ by setting $f(r) = 1$ if $r = 1$ and $f(r) = (r/p)^{1/p}$ if $r > 1$, where $p$ is the smallest member of $\pi$. Observe that if $r$ and $s$ are $\pi$-numbers, then $f(r)f(s) \leq f(rs)$. Expressed in terms of the function $f$, our goal is to find an element $x \in H$ such that $|C_N(x)|$ is a $\pi$-number and $|C_N(x)| \leq f(k)$.

Work by induction on $|H|$, and suppose first that it is possible to write $H = A \times B$, where $A$ and $B$ are proper subgroups of $H$ that admit the action of $N$. Let $M = C_N(A)$ and note that $M$ acts faithfully on $B$ since $N$ is faithful on $H$. Observe that $\pi$ contains all primes for which a Sylow subgroup of $M$ is nonabelian, and write $s = |M|_\pi$. Since $|B| < |H|$, the inductive hypothesis guarantees that there exists an element $b \in B$ such that $v = |C_M(b)|$ is a $\pi$-number with $v \leq f(s)$. 

Also, \( \pi \) contains all primes for which a Sylow subgroup of \( N/M \) is nonabelian, and we write \( r = |N/M|/\pi \), so that \( rs = k \). Since \( |A| < |H| \), we can apply the inductive hypothesis to the faithful action of \( N/M \) on \( A \), and we deduce that there exists an element \( a \in A \) such that \( u = |C_{N/M}(a)| \) is a \( \pi \)-number with \( u \leq f(r) \). By Lemma 2.1, we know that \( |C_{N}(ab)| \) divides \( uv \), and hence it is a \( \pi \)-number not exceeding \( uv \leq f(r)f(s) \leq f(rs) = f(k) \), as desired.

We can now assume that there is no proper decomposition \( H = A \times B \), where \( A \) and \( B \) admit \( N \). Let \( z \) be any nonidentity \( \pi \)-element of \( N \) and note that \( z \in \mathbf{Z}(N) \), so that both \( C_{H}(z) \) and \( [H, z] \) are \( N \)-invariant. By Fitting’s lemma, we have \( H = C_{H}(z) \times [H, z] \), and hence one of these factors must be trivial. Because the action of \( N \) on \( H \) is faithful, we cannot have \([H, z] = 1\), and thus \( C_{H}(z) = 1 \) for every nonidentity \( \pi \)-element \( z \) of \( N \). In other words, \( C_{N}(x) \) is a \( \pi \)-group for every nonidentity element \( x \in H \).

Let \( K \) be the Hall \( \pi \)-subgroup of \( N \) and note that \( |K| = k \). Since \( K \) acts faithfully on \( H \), we can apply Theorem 2 to this action to find a nonidentity element \( x \) in \( H \) such that \( |C_{K}(x)| \leq f(k) \). But \( C_{N}(x) \) is a \( \pi \)-group, and thus \( C_{N}(x) = C_{K}(x) \), and the result follows.

3. Centralizer chains

Instead of proving Theorem B directly, we prove a somewhat stronger result that lends itself to an inductive argument. To state this theorem, however, we need a somewhat technical definition. Let \( N \) act on \( H \) via automorphisms and write \( \mathcal{C}(N, H) \) to denote the collection of subgroups of \( N \) of the form \( C_{N}(X) \) for \( N \)-invariant subsets \( X \subseteq H \). (Observe that that because we require that the subsets \( X \) must be \( N \)-invariant, it follows that each member of \( \mathcal{C}(N, X) \) is a normal subgroup of \( N \).) We say that a totally ordered subset of \( \mathcal{C}(N, H) \) is a centralizer chain, and if \( C_{0} > C_{1} > \cdots > C_{r} \) is such a chain, we say that \( r \) is its length. Finally, we write \( r(N, H) \) to denote the maximum of the lengths of all centralizer chains for the action of \( N \) on \( H \). Note that if the action of \( N \) is nontrivial, then \( C_{N}(1) > C_{N}(H) \), and this is a centralizer chain of length 1. Thus \( r(N, H) \geq 1 \) for all nontrivial actions.

Observe that \( r(N, H) \) depends only on the permutation action of \( N \) on the elements of \( H \), and it is independent of the group structure of \( H \). It follows that if we use the Hartley-Turull lemma to replace \( H \) by an abelian group, this would not change the value of \( r(N, H) \).

The promised stronger form of Theorem B is the following.

(3.1) Theorem. Let \( N \) be a nilpotent \( \pi \)-group that acts faithfully on a \( \pi' \)-group \( H \). Then there exists \( x \in H \) such that

\[
|C_{N}(x)| \leq \left( \frac{|N|}{p'} \right)^{1/p},
\]

where \( p \) is the smallest member of \( \pi \) and \( r = r(N, H) \).

Note that if we apply Theorem 3.1 in the situation of Theorem B, where \( N \) is nontrivial, we have \( r = r(N, H) \geq 1 \), and thus there exists \( x \in H \) such that \( |C_{N}(x)| \leq (|N|/p')^{1/p} \). In other words, Theorem B really is a consequence of Theorem 3.1.

We begin to work now toward a proof of Theorem 3.1 with an easy lemma that shows how we will use the parameter \( r(N, H) \).
(3.2) Lemma. Let N act on H, and write \( r = r(N, H) \). Suppose that \( \mathcal{X} \) is a collection of \( N \)-invariant subsets of \( H \), and assume that \( \mathcal{X} \) is minimal with the property that the action of \( N \) on the subset \( \bigcup \mathcal{X} \) is faithful. Then \( |\mathcal{X}| \leq r \).

Proof. Write \( \mathcal{X} = \{X_1, X_2, \ldots, X_t\} \) and define

\[
C_i = C_N(\bigcup_{j=1}^{i} X_i)
\]

for \( 0 \leq i \leq t \), where we set \( C_0 = N \). The subgroups \( C_i \) all lie in \( C(N, H) \), and we have

\[
C_0 \supseteq C_1 \supseteq C_2 \cdots \supseteq C_t.
\]

If \( t > r \), then these containments cannot all be strict, and thus \( C_{j-1} = C_j \) for some integer \( j \) with \( 1 \leq j \leq t \). In other words, any element of \( N \) that centralizes each of the sets \( X_i \) for \( i < j \) also centralizes \( X_j \). This contradicts the minimality of the collection \( \mathcal{X} \), however, because it shows that \( X_j \) can be deleted, and \( N \) will act faithfully on the union of the resulting subcollection. It follows that \( t \leq r \), as required.

4. Proof of the main result

In this section we prove Theorem 3.1, and thereby complete the proof of Theorem B. We need the following numerical result.

(4.1) Lemma. Let \( x \geq y \geq 1 \). Then \( y^{x-1} \geq x^{y-1} \).

Proof. It suffices to show that \( (x - 1) \ln(y) \geq (y - 1) \ln(x) \). Now hold \( y \) fixed and view both sides of this inequality as functions of \( x \) as \( x \) varies over the range \( y \leq x < \infty \). Since equality holds when \( x = y \), it suffices to show that the derivative (with respect to \( x \)) of the left side is at least as large as the derivative of the right side. In other words, it suffices to show that \( \ln(y) \geq (y - 1)/x \) when \( y \) lies in the interval \( [1, x] \). Since equality holds when \( y = 1 \), it suffices to hold \( x \) fixed and compare the derivatives of both sides with respect to \( y \). What we want, therefore, is \( 1/y \geq 1/x \). This is true, of course, since \( y \leq x \).

Proof of Theorem 3.1. By the Hartley-Turull lemma, we can assume that \( H \) is abelian, and we proceed by double induction, first on \( |N| \) and then on \( |H| \). If \( N \) is trivial, then \( r = r(N, H) = 0 \), and the desired inequality holds. We can assume, therefore, that \( N > 1 \), and thus \( r \geq 1 \).

Suppose first that \( r > 1 \) and consider a centralizer chain \( C_r > C_{r-1} > \cdots > C_0 \) of length \( r \). Write \( M = C_1 \), and let \( A = C_H(M) \) and \( B = [H, M] \), so that \( H = A \times B \) by Fitting’s lemma. Also, \( A \) and \( B \) are \( N \)-invariant since \( M \triangleleft N \). Furthermore, if \( 1 \leq i \leq r \), we know that \( M \subseteq C_i = C_N(X_i) \) for some subset \( X_i \subseteq H \), and thus \( X_i \subseteq C_i \). Each of the subgroups \( C_i \), therefore, lies in \( C(N, A) \), and it follows that \( r(N/M, A) = r(N, A) \leq r - 1 \). Also, since \( r > 1 \) by assumption, we have that \( C_0 < M < C_r \), and thus \( 1 < M < N \). Finally, since \( X_1 \subseteq A \), it follows that \( M \) is the full kernel of the action of \( N \) on \( A \), and hence \( N/M \) acts faithfully on \( A \).

Since \( M \) centralizes \( A \), it must act faithfully on \( B \), and thus \( r(M, B) \geq 1 \) because \( M \) is nontrivial. Furthermore, \( |M| < |N| \), and so by the inductive hypothesis, there exists an element \( b \in B \) such that \( v = |C_M(b)| \) is small. In fact, we can choose \( b \) so that \( v \leq (|M|/p)^{1/p} \), where \( p \) is the smallest member of \( \pi \).
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Also, $N/M$ acts faithfully on $A$ and $|N/M| < |N|$, and so the inductive hypothesis applies to this action too. Since $r(N/M, A) \geq r - 1$, we can choose $a \in A$ and set $u = |C_{N/M}(a)|$ so that $u \leq (|N/M|/p^{r-1})^{1/p}$. By Lemma 2.1, we have

$$|C_N(ab)| \leq uw \leq \left(\frac{|N/M|}{p^{r-1}}\right)^{1/p} \left(\frac{|M|}{p^r}\right)^{1/p} = \left(\frac{|N|}{p^r}\right)^{1/p},$$

and the proof is complete in this case.

We can now assume that $r = 1$, and so $C(N, H) = \{1, N\}$, and our goal in this case is to find $x \in H$ such that $|C_N(x)| \leq (|N|/p)^{1/p}$. Let $H/A$ be an $N$-composition factor of $H$, and note that $C_N(A) \in C(N, H)$. If $C_N(A) = 1$, then $N$ acts faithfully on $A$, and since $A < H$, the result follows by the inductive hypothesis on $|H|$. The only other possibility is that $C_N(A) = N$, and hence $A = C_H(N)$ and we can write $H = A \times B$, where $B = [H, N]$, and $N$ acts faithfully on $B$. Again, we are done by the inductive hypothesis if $B < H$, and so we can assume that $B = H$ and $A = 1$. In other words, we are reduced to the case where $H$ is a simple $N$-module in characteristic not dividing $|N|$.

Suppose now that every abelian normal subgroup of $N$ is cyclic. Since $N$ is nilpotent, every Sylow subgroup of $N$ satisfies the same hypothesis, and it follows that the odd Sylow subgroups are cyclic and that the Sylow 2-subgroup of $N$ is either cyclic, or else is dihedral, semidihedral or generalized quaternion. In any case, we see that $N$ has a cyclic subgroup $C$ of index not exceeding 2.

If $1 \neq x \in H$, then $C_C(x)$ is characteristic in $C$, and hence is normal in $N$. It follows that $C_H(C_C(x))$ is a nonidentity $N$-invariant subgroup of $H$, which must, therefore, be all of $H$. Thus $C_C(x)$ acts trivially on $H$, and we conclude that $C_C(x) = 1$, and hence $C \cap C_N(x) = 1$. Therefore $|C_N(x)| \leq |N : C| \leq 2$. If $C = N$, then $|C_N(x)| = 1 \leq (|N|/p)^{1/p}$, as required. Otherwise, $|N|$ is even, and hence $p = 2$. In this case, $|N| \geq 8$ and we have $|C_N(x)| \leq 2 \leq (|N|/2)^{1/2}$, and we are done in this case also.

Finally, we can suppose that $N$ has a noncyclic abelian normal subgroup, and hence it is imprimitive in its action on the simple module $H$. It is therefore possible to write $H$ as a nontrivial direct sum of a collection $X$ of subgroups transitively permuted by the action of $N$. Choosing $X$ so that $|X| = q$ is as small as possible, we note that $q$ is the index in $N$ of the stabilizer $R$ of one of the members of $X$. Also, $R$ is a maximal subgroup of $N$, and hence $R \lhd N$, and thus $R$ stabilizes all of the subgroups in $X$. Every subgroup of $N$ not contained in $R$ therefore acts transitively on $X$. Also, $R > 1$ since $N$ is noncyclic.

Suppose first that $r(R, H) < q$. By Lemma 3.2, it follows that $R$ acts faithfully on the sum $K$ of $q - 1$ of the members of $X$. Since $K < H$ and $r(R, K) \geq 1$ because $R > 1$, the inductive hypothesis applies, and we can find $x \in K$ such that $|C_R(x)| \leq (|R|/p)^{1/p}$. Because $x \in K$, the component of $x$ in at least one of the direct summands in $X$ is trivial, and yet $x$ is nontrivial. It follows that $C_N(x)$ cannot permute the set $X$ transitively. Thus $C_N(x) \subseteq R$ and $|C_N(x)| = |C_R(x)| \leq (|R|/p)^{1/p} \leq (|N|/p)^{1/p}$, as required.

We can now assume that $r(R, H) \geq q$, and thus by the inductive hypothesis (on $|N|$), we can find $x \in H$ such that $|C_R(x)| \leq (|R|/p^q)^{1/p}$. Since $|C_N(x)| \leq |N : R|\cdot |C_R(x)|$ and $|N : R| = q$, we have

$$|C_N(x)|^p \leq q^p|C_R(x)|^p \leq q^p \left(\frac{|R|}{p^q}\right)^{1/p} = \frac{q^p|N|}{qq^q} = \left(\frac{|N|}{p}\right) \left(\frac{p}{q^q-1}\right) \leq \frac{|N|}{p},$$

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where the final inequality follows from Lemma 4.1 since $p \leq q$. The result now follows.

5. Further remarks

We have no evidence that the hypothesis that $N$ is nilpotent is really necessary in Theorem B. It seems that if $N$ acts coprimely and faithfully on $H$, it should at least be true that there exists an element $x \in H$ such that $|C_N(x)| \leq |N|^{1/2}$, but we are unable to to prove this even if $N$ is assumed to be solvable.

In the case where the acting group $N$ is abelian, we can take $\pi$ to be empty in Theorem C, and we deduce that there exists $x \in H$ such that $C_N(x) = 1$. It is easy to prove this directly by using the Hartley-Turull lemma to replace $H$ by an abelian group, but there is another elementary proof available that is perhaps not as well known as it deserves to be. It is possible to obtain the result by using a variation on Brodkey’s theorem [1], applied to the semidirect product $G = HN$.

Every conjugate of $N$ in $G$ has the form $N^x$ with $x \in H$, and thus it suffices to find such a conjugate that intersects $N$ trivially. Note that Brodkey’s theorem applies since $G$ is $\pi$-separable, and so every subgroup satisfies property $D_\pi$. Also, $O_\pi(G) = 1$ because $N$ acts faithfully on $H$.

(5.1) Theorem (Brodkey). Let $N$ be an abelian Hall $\pi$-subgroup of a group $G$ and assume that every subgroup of $G$ satisfies property $D_\pi$. If $O_\pi(G) = 1$, then $N \cap M = 1$ for some $G$-conjugate $M$ of $N$.

Proof. Choose a conjugate $M$ of $N$ such that $D = M \cap N$ is minimal, and assume that $D > 1$. Since $\text{core}_G(N) = 1$, we can choose some conjugate $Q$ of $N$ such that $D \not\subseteq Q$. Now let $U = \langle N, M \rangle$ and note that $D \triangleleft U$ since $M$ and $N$ are abelian. Also, for some element $u \in U$, we have $(U \cap Q)^u \subseteq M$ by the $D_\pi$ property in the subgroup $U$. Then $N \cap Q^u = N \cap U \cap Q^u \subseteq N \cap M = D$. By the minimality of $D$, we deduce that $N \cap Q^u = D$, and thus $D^u = D \subseteq Q^u$. This is a contradiction, however, because $D \not\subseteq Q$.

REFERENCES


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