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STRUCTURAL STABILITY ON BASINS FOR NUMERICAL METHODS

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ABSTRACT. In this paper, we show that a flow φ with a hyperbolic compact attracting set is structurally stable on the basin of attraction with respect to numerical methods. The result is a generalized version of earlier results by Garay, Li, Pugh, and Shub. The proof relies heavily on the usual invariant manifold theory elaborated by Hirsch, Pugh, and Shub (1977), and by Robinson (1976).

1. STATEMENT OF THEOREMS

Let M be a smooth complete Riemannian manifold with a distance d arising from the Riemannian metric and Diff(M) be the set of diffeomorphisms on M with the strong topology and distance d_{C^1}. A *flow* is a map $\varphi : \mathbb{R} \times M \to M$ that satisfies the group property: $\varphi^s(\varphi^t(x)) = \varphi^{s+t}(x)$.

A set $\mathcal{A}$ is *attracting* for a flow φ on M if there is a neighborhood U of $\mathcal{A}$ such that $\varphi^T(cl(U)) \subset int(U)$ for some $T > 0$ and $\mathcal{A} = \bigcap_{t\geq 0} \varphi^t(U)$. The *basin* of $\mathcal{A}$ is the set $B(\mathcal{A}) = \{x \in M : \lim_{t\to\infty} d(\varphi^t(x), \mathcal{A}) = 0\}$, where $d(y, \mathcal{A}) = \min\{d(y, z) : z \in \mathcal{A}\}$. An attracting set for a flow is closed and invariant (see [18]).

A compact invariant set $\mathcal{A}$ for a flow φ on M is *hyperbolic* if the restriction of the tangent bundle TM of M to $\mathcal{A}$ splits into three continuous subbundles, $TM|\mathcal{A} = \mathbb{E}^u \oplus \mathbb{E}^s \oplus Span(X)$, invariant under the derivative of φ^t, $D\varphi^t$, such that $D\varphi^t$ expands $\mathbb{E}^u$ and $D\varphi^t$ contracts $\mathbb{E}^s$. Here X is the vector field induced by the flow φ.

Definition 1. For $p \geq 1$, let φ be a C^{p+1} flow on M. A C^{p+1} function N: $\mathbb{R} \times M \to M$ is called a *numerical method of order* p for φ if there are positive constants K and h_0 such that $d(\varphi^h(x), N^h(x)) \leq Kh^{p+1}$, for all $h \in [0, h_0]$ and $x \in M$. Here h stands for a *stepsize* of N. We denote the i-th iterate of $N^h(x)$ by $(N^h)^i(x)$.

Numerical methods arise from computer simulation and numerical approximation. For instance, both explicit and implicit Runge-Kutta methods are of order $p \geq 4$ (see [1]).

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It is well known that the time-h map of the flow and the numerical method of stepsize h are C^1 close exponentially in terms of h.

Lemma 1 ([7]). *Let N be a numerical method of order p for a C^{p+1} flow φ on a compact manifold M. Then there is a constant K_1 such that $d_{C^1}(\varphi^h, N^h) \le K_1 h^p$ for all sufficiently small h. Moreover, given $T > 0$, there is a constant K_2 such that $d_{C^1}(\varphi^T, (N^{\frac{T}{n}})^n) \le K_2 n^{1-p}$ for all large $n \in \mathbb{N}$.*

Now we state our main result.

Theorem 1. *Let $p \ge 2$, $\mathcal{A}_\varphi$ be a hyperbolic attracting set for a C^{p+1} flow φ on a compact manifold M, and $B(\mathcal{A}_\varphi)$ be the basin of $\mathcal{A}_\varphi$. Let N be a numerical method of order p for φ and $T > 0$ be given. If n is sufficiently large, then there is a homeomorphism H_n from $B(\mathcal{A}_\varphi)$ to its image and a continuous function $\tau_n : B(\mathcal{A}_\varphi) \to \mathbb{R}$ such that for all $x \in B(\mathcal{A}_\varphi)$,*

$$H_n \circ \varphi^{\tau_n(x)}(x) = (N^{\frac{T}{n}})^n \circ H_n(x).$$

Theorem 1 does not work for the Euler method because it is of order $p = 1$ (see [1]). We discuss this case separately. In order to describe the Euler method on an abstract manifold, we need a local chart. As in [19], let $\Psi_x : U_x \to M$ be a function on a neighborhood U_x of 0 in T_xM such that $\Psi_x(0) = x$, $D\Psi_x(0) = id_{T_xM}$, and $\Psi : U \to M$ defined by $(x, v) \to \Psi_x(v)$ is smooth with Lipschitz first and second derivative which are bounded along M.

Definition 2. Let X be a vector field on M and φ be the flow of the differential equation $\dot{x} = X(x)$. For $h > 0$ small, let $E^h(x) = \Psi_x(hX(x))$. We call E^h the *Euler method of stepsize h* for φ.

This agrees with the usual definition in Euclidean space where $\Psi_x(X(x)) = x + X(x)$ and $E^h(x) = x + hX(x)$.

Fortunately, we also have the C^1 closeness between the time-h map of the flow and the Euler method of stepsize h. The following lemma was given by Shub [19].

Lemma 2. *If X is a C^2 bounded vector field on M, φ is the flow of the differential equation $\dot{x} = X(x)$, and E is the Euler method for φ. Then for all sufficiently small h, there is a positive constant K_1 such that $d_{C^1}(\varphi^h, E^h) \le K_1 h^2$. Moreover, given $T > 0$, there is a positive constant K_2 such that $d_{C^1}(\varphi^T, (E^{\frac{T}{n}})^n) \le K_2 n^{-1}$ for all large $n \in \mathbb{N}$.*

Omitting the same argument as in the proof of Theorem 1, we only state the result for the Euler method.

Theorem 2. *Let X be a C^2 vector field on a compact manifold M such that the differential equation $\dot{x} = X(x)$ induces a flow φ with a hyperbolic attracting set $\mathcal{A}_\varphi$. Let E^h be the Euler method with stepsize h for φ and $T > 0$ be given. Then for all sufficiently large n, there is a homeomorphism H_h from $B(\mathcal{A}_\varphi)$ to its image and a continuous function $\tau_n : B(\mathcal{A}_\varphi) \to \mathbb{R}$ such that for all $x \in B(\mathcal{A}_\varphi)$,*

$$H_n \circ \varphi^{\tau_n(x)}(x) = (E^{\frac{T}{n}})^n \circ H_n(x).$$

Our theorems fit well in a list of results on stability for numerical methods. Garay [8] asserted that the restriction of φ to $B(\mathcal{A}_\varphi)\backslash\mathcal{A}_\varphi$, $\varphi|B(\mathcal{A}_\varphi)\backslash\mathcal{A}_\varphi$, is structurally stable under discretization. Beyn [2] proved that hyperbolic fixed points persist under numerical methods and the whole saddle-point structure in a neighborhood of a hyperbolic fixed point is preserved. Garay [6] showed that on a neighborhood of a hyperbolic fixed point, the time-h map of the flow and a numerical method of stepsize h can be conjugated by a local homeomorphism. Fečkan [5] proved the very same result in the case of the Euler method. See also [9] and [13] for structural stability of Morse-Smale gradient-like flows under numerics. Pugh and Shub [14] showed that a hyperbolic periodic orbit persists as an invariant embedded circle under solution schemes. Special cases were discussed in [2], [3] and [4]. See [14] and [7] for the persistence of normally hyperbolic invariant manifold and also [12] and [11] for the persistence of attractors.

2. Proof of Theorem 1

We closely follow the proofs of abstract invariant manifold theorems in [10] and [17]. We first outline the proof and then present the details.

We want to construct the conjugacy H_n on the basin of attraction $B(\mathcal{A}_\varphi)$. The usual approach is to consider a C^1 bundle map F defined on a fixed smooth normal bundle η of the flow φ contained in TM and then to construct unstable manifolds for F of the zero section of $\eta|U$, where U is a small neighborhood of $\mathcal{A}_\varphi$. Following the methods in [10] and [17], we consider trial unstable manifolds which are graphs of sections $\sigma : \eta^u(r)|U \to \eta$, where $\eta|U = \eta^u \oplus \eta^s$ is an almost invariant splitting and $\eta^u(r)$ is the disk bundle of radius r. The graph transform by F, $F_\#(\sigma)$, is defined over $\varphi^T(U)$ in such a way that $F(image(\sigma)) \supset image(F_\#(\sigma))$. We need to extend $F_\#(\sigma)$ back over a fundamental domain $V = U \backslash \varphi^T(U)$. We first construct a section over V, $\sigma_0 : \eta^u(r)|V \to \eta$, whose graph is an invariant manifold for F of the zero section. We then only consider sections σ that agree with σ_0 over V, and extend $F_\#(\sigma)$ to V by means of σ_0. By results of [10] and [17], we get that the graph transform $F_\#$ is a contraction on a space of sections. Hence there is a unique fixed point $\sigma^{u\varphi} : \eta^u(r)|U \to \eta$ such that $F_\#(\sigma^{u\varphi}) = \sigma^{u\varphi}$, and $\sigma^{u\varphi} = \sigma_0$ over V. The unstable manifolds for F of the zero section of $\eta|U$, $Z^{u\varphi}(x) = image(\sigma^{u\varphi}_x)$, are contained in η. We get what is called in [17] a family of *unstable disks* for φ^T, $\{Z^{u\varphi}(x) : x \in U\}$.

To prove structural stability, for $(N^{\frac{T}{n}})^n$ near φ^T we consider a bundle map G_n is C^1 near F. Again, we get families of stable and unstable disks for G_n that are continuous and near those for F. Their intersection gives a section $v_n : U \to \eta$ that is continuous. Then $H_n \equiv \exp v_n$ is a semiconjugacy of φ^T and $(N^{\frac{T}{n}})^n$. By the expansiveness of φ on $\mathcal{A}_\varphi$ and the way we construct σ_0 over V, we can show H_n is one-to-one on $B(\mathcal{A}_\varphi)$.

We divide the proof into several steps.

Step 1: Preliminary setup.

Because $\mathcal{A}_\varphi$ is hyperbolic, the tangent bundle of M along $\mathcal{A}_\varphi$ splits as the sum of three bundles $TM|\mathcal{A}_\varphi = \mathbb{E}^u \oplus \mathbb{E}^s \oplus Span(X)$, where $X(x) = \frac{d\varphi^t}{dt}(x)|_{t=0}$ is the tangent vector field for φ. Let U_0 be a small neighborhood of $\mathcal{A}_\varphi$. We want the normal bundle η of φ to be smooth. It is no loss of generality to make a convenient choice of η: Let η^u and η^s be smooth subbundles of $TM|U_0$ with approximating $\mathbb{E}^u$ and $\mathbb{E}^s$ so that $TM|U_0 = \eta^u \oplus \eta^s \oplus Span(X)$, and choose $\eta = \eta^u \oplus \eta^s$. To make the analysis easier, we assume that the bundles η^u and η^s can be trivialized, i.e., for $\delta = u, s$, we assume there exists a C^1 bundle B^δ over U_0 such that $\eta^\delta \oplus B^\delta = U_0 \times Y^\delta$, where Y^δ is a Banach space, and the Finsler (continuous norm) on η^δ is induced

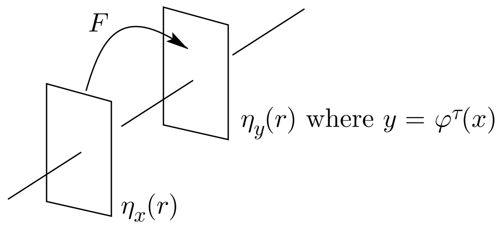

FIGURE 1

from the norm on Y^δ. Let $\eta^\delta(r) = \{v \in \eta^\delta : |v| \le r\}$, $\delta = u, s$, be the r disk bundles and $\eta(r) = \eta^u(r) \oplus \eta^s(r)$.

To get a grip on the space of sections, we need to define a section's slope. If $\sigma : \eta^u(r) \to \eta$ is a section, then the *slope* of σ at $v_x \in \eta^u(r)$ is

$$\limsup_{v_y \to v_x} \frac{|s(v_x) - s(v_y)|_{Y^s}}{d_u(v_x, v_y)},$$

where $\sigma(v_x) = (v_x, s(v_x)) \in \eta^u \times Y^s$, $|\cdot|_{Y^s}$ is the norm on Y^s, and d_u is the Finsler metric on η^u. Let $\Sigma(1,r) = \{\text{section } \sigma : \eta^u(r) \to \eta(r) \text{ such that } \mathit{slope}(\sigma) \le 1\}$. Putting the C^0 sup norm on $\Sigma(1,r)$ makes it a complete metric space as usual. Let $\pi^u : \eta \to \eta^u$ be a projection along η^s and $\pi^s : \eta \to \eta^s$ be a projection along η^u.

Step 2: Definition of bundle map.

We use the concept of laminations in [10] to define a bundle map F on η. For f near φ^T in the C^1 topology, let $\Theta(\tau, v_x, f) = \exp_y^{-1} \circ f \circ \exp v_x$ where $y = \varphi^\tau(x)$. There is a neighborhood $U_1 \subset U_0$ of $\mathcal{A}_\varphi$, a constant $r_1 > 0$, a neighborhood $\mathcal{U}$ of φ^T in $\mathrm{Diff}(M)$, and a continuous function $\tau : \eta(r_1)|_{U_1} \times \mathcal{U} \to \mathbb{R}$ such that for all $x \in U_1$, $v_x \in \eta_x(r_1)$, and $f \in \mathcal{U}$,

$$\Theta(\tau(v_x, f), v_x, f) \in \eta_{\varphi^{\tau(v_x,f)}(x)}(r_1).$$

Here τ stands for a reparameterization of φ. See page 95 of [10] and also [16]. Define a bundle map F by

$$F(v_x) \equiv \Theta(\tau(v_x, \varphi^T), v_x, \varphi^T) = \exp^{-1}_{\varphi^\tau(x)} \circ \varphi^T \circ \exp v_x.$$

Then F is a C^1 bundle map on $\eta(r_1)$. See Figure 1.

Step 3: Definition of graph transform.

To define the graph transform of F, we need an inverse of $\pi^u \circ F \circ \sigma$ for $\sigma \in \Sigma(1,r)$. As in page 102 of [10], one can take $r_2 \le r_1$ small so that for $\sigma \in \Sigma(1, r_2)$ and $x \in \mathcal{A}_\varphi$, $\pi^u \circ F \circ \sigma : \eta^u(r_2)|_{O(x)} \to \eta^u|_{O(x)}$ is invertible, where $O(x)$ is the trajectory of x under the flow φ. Let $U_2 \subset U_1$ be a small neighborhood of $\mathcal{A}_\varphi$ such that $\pi^u \circ F \circ \sigma$ is invertible on $\eta^u(r_2)|_{U_2}$ and has an inverse g defined on $\eta^u(r_2)|_{\varphi^T(U_2)}$, $g : \eta^u(r_2)|_{\varphi^T(U_2)} \to \eta^u(r_2)$.

In order to construct the conjugacy not only on the attractor but also on the basin of attraction, we adapt the method of Robinson [17], see also [15]. Let $U \subset U_2$ be a small neighborhood of $\mathcal{A}_\varphi$ so that $V = \mathit{closure}(U \backslash \varphi^T(U))$ is a proper fundamental domain. Here $\varphi^T(U) \subset U$ because $\mathcal{A}_\varphi$ is an attracting set. Let β be a bump function that equals zero near the exterior boundary of V, or ∂U, and equals 1 near the interior boundary of V, or $\partial \varphi^T(U)$. Let $\sigma_1 : \eta^u(r_2) \to \eta$ be defined by $\sigma_1(v) = v$, i.e. $\pi^s \sigma_1(v) = 0$, and $\sigma_2 = F \circ \sigma_1 \circ g$. Define $\sigma_0 : \eta^u \to \eta$ by $\pi^s \sigma_0(v_x) = \beta(x) \pi^s \sigma_2(v_x)$.

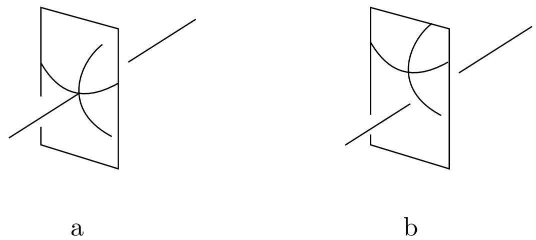

FIGURE 2

Now we define a graph transform $F_{\#}$ of F over U by

$$F_{\#}(\sigma)_x = \begin{cases} F \circ \sigma \circ g_x & \text{for } x \in \varphi^T(U), \\ \sigma_{0x} & \text{for } x \in V. \end{cases}$$

Let $\Sigma(1, r, \sigma_0) = \{$section $\sigma : \eta^u(r) \to \eta(r)$ such that $\sigma = \sigma_0$ on the domain of σ_0 and $slope(\sigma) \le 1\}$. By Theorem 6.1 of [10] and Theorem 3.1 of [17], we get that $F_{\#}$ is a contraction on $\Sigma(1, r_2, \sigma_0)$ and has a unique fixed point $\sigma^{u\varphi} \in \Sigma(1, r_2, \sigma_0)$.

Step 4: Construction of conjugacy and reparameterization.

In order to prove structural stability, we do the same construction for $(N^{\frac{T}{n}})^n$. By Lemma 1, we have $(N^{\frac{T}{n}})^n \to \varphi^T$ in the C^1 topology as $n \to \infty$. Take n sufficiently large so that $(N^{\frac{T}{n}})^n \in \mathcal{U}$, the neighborhood of φ^T in $\mathrm{Diff}(M)$ where Θ is well-defined. Let

$$G_n(v_x) \equiv \Theta(\tau(v_x, (N^{\frac{T}{n}})^n), v_x, (N^{\frac{T}{n}})^n) = \exp^{-1}_{\varphi^\tau(x)} \circ (N^{\frac{T}{n}})^n \circ \exp v_x.$$

Then G_n is C^1 bundle map on $\eta(r_1)$ and C^1 close to F. We define σ^N_{n0} over V by $\sigma^N_{n0}(v_x) = \beta(x)(N^{\frac{T}{n}})^n \circ \sigma_1 \circ g(v_x)$. Because $(N^{\frac{T}{n}})^n$ is C^1 near φ^T, σ^N_{n0} is C^1 near σ_0. By the permanence results in Theorem 3.1 of [17] and Theorem 6.1 of [10], the graph transform of G_n, $G_{n\#}$, is C^1 close to $F_{\#}$, carries $\Sigma(1, r_2, \sigma^N_{n0})$ into itself, and is a contraction. Therefore, $G_{n\#}$ has a unique fixed point $\sigma^{uN}_n \in \Sigma(1, r_2, \sigma^N_{n0})$.

Because φ^{-T} is overflowing on U, without defining σ_0 on a fundamental domain, we can construct $\sigma^{s\varphi}$ and σ^{sN}_n. The disks $Z^u(x, \varphi) = image(\sigma^{u\varphi}_x)$ and $Z^s(x, \varphi) = image(\sigma^{s\varphi}_x)$ intersect transversally at the zero vector, see Figure 2a. The disks $Z^u_n(x, N) = image(\sigma^{uN}_{nx})$ and $Z^s_n(x, N) = image(\sigma^{sN}_{nx})$ intersect transversally in each fiber at a unique point, $v_n(x)$, see Figure 2b. Then $v_n : U \to \eta(r_2)$ is a continuous section and $G_{n\#} v_n = v_n$. We extend v_n to the basin of attraction, $B(\mathcal{A}_\varphi)$.

We now define the conjugacy and the reparameterization. Let $H_n(x) = \exp v_n(x)$ and $\tau_n(x) = \tau(v_n(x), (N^{\frac{T}{n}})^n)$ for all $x \in B(\mathcal{A}_\varphi)$. Then $H_n : B(\mathcal{A}_\varphi) \to M$ and $\tau_n : B(\mathcal{A}_\varphi) \to \mathbb{R}$ are continuous. Moreover $v_n \circ \varphi^{\tau_n(x)}(x) = \exp^{-1}_{\varphi^{\tau_n(x)}(x)} \circ (N^{\frac{T}{n}})^n \circ \exp v_n(x)$ and so $H_n \circ \varphi^{\tau_n(x)}(x) = (N^{\frac{T}{n}})^n \circ H_n(x)$. It remains to show that H_n is one-to-one.

Step 5: Prove H_n is one-to-one.

Assume that $H_n(x) = H_n(y)$. Then $H_n \circ \varphi^{\tau_n(x)}(x) = (N^{\frac{T}{n}})^n \circ H_n(x) = (N^{\frac{T}{n}})^n \circ H_n(y) = H_n \circ \varphi^{\tau_n(y)}(y)$. There exist $\{x_i\}$ and $\{y_i\}$ lying on the flow trajectories of x and y, respectively, so that $H_n(x_i) = ((N^{\frac{T}{n}})^n)^i(H_n(x)) = ((N^{\frac{T}{n}})^n)^i(H_n(y)) = H_n(y_i)$ as long as $x_i, y_i \in B(\mathcal{A}_\varphi)$. Therefore $\exp v_n(x_i) = \exp v_n(y_i)$ and $d(x_i, y_i) \le$

$2r_2$ for all $i \geq 0$. Suppose $x \in \mathcal{A}_\varphi$. By the flow expansiveness of φ at $\mathcal{A}_\varphi$ (see [18]), the points x and y must lie in the same flow trajectory. But transversal disks are disjoint for two nearby points on the same trajectory. Thus $x = y$. This proves H_n is one-to-one on $\mathcal{A}_\varphi$.

Suppose $x \notin \mathcal{A}_\varphi$, then take i_0 so that $x_{i_0} \in V$ and y_{i_0} is near V. It is still true that $\{y_i : i \geq 0\}$ stays near the forward trajectories of $\{x_{i_0}\}$ under φ^t, and so $y_{i_0} \in W^s(O(x_{i_0}), \varphi^t)$, the stable manifold of the orbit of x_{i_0}. On V, the unstable disks $Z^u(z, N)$ are uniformly C^1 and transverse to the stable direction $W^s(O(x_{i_0}), \varphi^t)$. Thus the unstable disks form a tubular neighborhood of $W^s(O(x_{i_0}), \varphi^t)$. On the other hand, the fact that $\exp v_n(x_{i_0}) = \exp v_n(y_{i_0})$ gives us $v_n(x_{i_0}) \in Z^u(x_{i_0}, N)$ and $v_n(y_{i_0}) \in Z^u(y_{i_0}, N)$, and so these two unstable disks intersect. Therefore, $x_{i_0} = y_{i_0}$ and we get that x and y are in the same flow trajectory. Again $x = y$ as above. This proves that H_n is one-to-one on $B(\mathcal{A}_\varphi)$.

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