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FINITELY GENERATED GROUP RING UNITS

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Abstract. We give a classification of nilpotent groups G for which the unit group of the integral group ring is finitely generated.

For a group G let $\mathcal{U}\mathbb{Z}G$ denote the group of invertible elements in the integral group ring $\mathbb{Z}G$. This is a subject of intensive study; see Sehgal [5].

The natural question arises as to when $\mathcal{U}\mathbb{Z}G$ is finitely generated. It is easy to see that if this is the case, then G must be finitely generated. Yet the question has remained unanswered even for the (simplest nonabelian) case of nilpotent groups. In the literature one can find a few partial results. The first is

1. Theorem (Sehgal [4], p. 194)**.** *Suppose that G is an extension of a torsion group T by a torsion free nilpotent group. If $\mathbb{Z}G$ has no nilpotent elements, then $\mathcal{U}\mathbb{Z}G = (\mathcal{U}\mathbb{Z}T)\cdot G$.* □

It is well known that $\mathcal{U}\mathbb{Z}T$ is finitely generated for any finite group T and hence Theorem 1 implies that $\mathcal{U}\mathbb{Z}G$ is finitely generated if $\mathbb{Z}G$ contains no nilpotent elements. On the other hand we have a result obtained recently by Marciniak and Sehgal [2] who proved that $\mathcal{U}\mathbb{Z}G$ is not finitely generated for $G = \mathbb{Z} \times D_8$, where D_8 is the dihedral group of order 8. Another result of this sort is due to Mirowicz [3]. He showed that $\mathcal{U}\mathbb{Z}[D_\infty]$ is not finitely generated for the infinite dihedral group D_∞.

In the present paper we determine when $\mathcal{U}\mathbb{Z}G$ is finitely generated for nilpotent groups G. In what follows we denote by $T(G)$ the set of torsion elements in G and by $\mathcal{Z}(G)$ the center of G. It is well known for finitely generated nilpotent groups G that $T(G)$ is a finite characteristic subgroup which is the product of its Sylow subgroups. For the details concerning nilpotent groups we refer the reader to [1]. Our main result reads:

2. Theorem. *Let G be a nilpotent group. Then $\mathcal{U}\mathbb{Z}G$ is finitely generated if and only if either G is finite or G is finitely generated and there are no nilpotent elements in $\mathbb{Z}G$.*

By using Sehgal's characterization of group rings which have no nilpotent elements (see Theorem 10) one can formulate this result in purely group theoretic terms as follows.

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3. Corollary. *Let G be a nilpotent group. Then $\mathcal{U}\mathbb{Z}G$ is finitely generated if and only if either G is finite or G is finitely generated, all finite subgroups of G are normal and its torsion part $T(G)$ satisfies one of the following conditions:*

(i) *$T(G)$ is abelian, or*

(ii) *$T(G) = A \times E \times K_8$ where E is an elementary abelian 2-group and A is a finite abelian group of odd order m such that the multiplicative order of $2 \bmod m$ is odd.*

We start with four auxiliary results.

4. Lemma. *Let G be a nilpotent group with a finite normal p-subgroup H. Then the ideal $\Delta(G,H) = \ker(\mathbb{F}_pG \to \mathbb{F}_p[G/H])$ is nilpotent.*

Proof. We proceed by induction on $|H|$. If $H = \langle t | t^p = 1\rangle$, then

$$\Delta(G,H) = (\mathbb{F}_pG)(t-1) = (t-1)(\mathbb{F}_pG)$$

and hence for any $\alpha_i = \beta_i(t-1) \in \Delta(G,H)$ $(1 \le i \le p)$ we have $\alpha_1 \cdots \alpha_p \in \mathbb{F}_pG(t-1)^p = \mathbb{F}_pG(t^p-1) = 0$. Therefore the lemma is true when $|H| = p$.

Suppose now $|H| = p^n$, $n > 1$. As G is nilpotent and $H \lhd G$, so $H \cap \mathcal{Z}(G) \neq 1$ and hence there exists a central subgroup $C \lhd H$ with $|C| = p$. For $\phi\colon \mathbb{F}_pG \to \mathbb{F}_p[G/C]$ we clearly have $\phi(\Delta(G,H)) \subseteq \Delta(G/C, H/C)$. By the inductive hypothesis, $\Delta(G/C, H/C)$ and $\Delta(G,C)$ are nilpotent, hence

$$(\Delta(G,H))^{st} \subseteq (\Delta(G,C))^t = 0$$

for suitable s, t. □

5. Lemma. *Let G be a finitely generated nilpotent group of Hirsch rank 1. Then G is a semidirect product of its finite subgroup $T(G)$ with an infinite cyclic group.*

Proof. $G/T(G)$ is a finitely generated torsion free nilpotent group of Hirsch rank 1, so it is infinite cyclic. □

6. Lemma. *If G is a finitely generated nilpotent group with an infinite and normal subgroup H, then there exists a subgroup $H_1 < H$, $H_1 \lhd G$, such that $H/H_1 \simeq \mathbb{Z}$.*

Proof. Consider the upper central sequence of G: $\zeta_0G < \zeta_1G < \zeta_2G < \cdots$ and the subgroups $D_j = \zeta_jG \cap H$. Let i be the (least) integer satisfying $|H/D_i| = \infty$ and $|H/D_{i+1}| < \infty$. Then D_{i+1}/D_i is infinite, finitely generated and lies in the center of G/D_i. Besides, $|(H/D_i) : (D_{i+1}/D_i)| < \infty$. Thus we can find a group $D_{i+\frac{1}{2}}$ such that $D_i \le D_{i+\frac{1}{2}} < D_{i+1}$ and $D_{i+1}/D_{i+\frac{1}{2}} \simeq \mathbb{Z}$. From Lemma 5 it follows that $H/D_{i+\frac{1}{2}}$ is a semidirect product of $\mathbb{Z}$ with its subgroup T of torsion elements which, by the way, is normal in $G/D_{i+\frac{1}{2}}$. Hence we may define

$$H_1 = \pi^{-1}(T)$$

where $\pi\colon G \to G/D_{i+\frac{1}{2}}$ is the natural homomorphism. □

7. Lemma. *Let R be a ring of characteristic p and J its nilpotent ideal. Then $1 + J$ is a nilpotent p-group.*

Proof. If $y \in J$, then $(1+y)^{-1} = 1 - y + y^2 - \cdots \in 1 + J$, hence $1 + J$ is a group. Next observe that $[1 + J^k, 1 + J^l] \subseteq 1 + J^{k+l}$. Indeed, if $a \in J^k$ and $b \in J^l$, then $[1-a, 1-b] = (1 + a + a^2 + \cdots)(1 + b + b^2 + \cdots)(1-a)(1-b) = 1+$ (a sum of products in which both a and b occur) $\in 1 + J^{k+l}$. So our inclusion holds and

consequently $1+J$ is a nilpotent group. To prove that $1+J$ is a p-group notice that if $a \in J$, then $(1+a)^{p^n} = 1 + a^{p^n} = 1$ for n sufficiently large. $\square$

Definition. A group ring unit $u \in \mathbb{Z}G$ is unipotent if $(u-1)^n = 0$ for some $n \geq 1$.

Definition. A subgroup $U < \mathcal{U}\mathbb{Z}G$ is p-nice if $\mathcal{Z}(G) < U$ and G has a finite normal p-subgroup V such that the image of U in $\mathbb{Z}[G/V]$ consists of trivial units only.

8. Theorem. *Suppose $U < \mathcal{U}\mathbb{Z}H$ is p-nice and the center $\mathcal{Z}(H)$ is of finite index in H. If U is finitely generated, then any infinite set of unipotent units in U has two elements u, v such that $p|(u-v)$.*

Proof. We have two commuting diagrams of natural maps:

$$\begin{array}{ccc} \mathbb{Z}H & \xrightarrow{\pi} & \mathbb{Z}[H/V] \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ \mathbb{F}_pH & \xrightarrow{\bar\pi} & \mathbb{F}_p[H/V] \end{array} \qquad \begin{array}{ccc} \mathcal{U}\mathbb{Z}H & \xrightarrow{\pi} & \mathcal{U}\mathbb{Z}[H/V] \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ \mathcal{U}\mathbb{F}_pH & \xrightarrow{\bar\pi} & \mathcal{U}F_p[H/V] \end{array}$$

Let $\bar U = \rho(U) < \mathcal{U}\mathbb{F}_pH$ and $I = \ker(\bar\pi\colon \mathbb{F}_pH \to \mathbb{F}_p[H/V]) = \Delta(H,V)$. As $\bar\pi(\bar U) = \rho\pi(U) < \pm H/V$, so we have an exact sequence

$$1 \to (1+I)\cap\bar U \to \bar U \to \pm H/V.$$

Let $Z = \mathcal{Z}(H)$. We obtain an exact sequence

$$1 \to ((1+I)\cap\bar U)/((1+I)\cap\bar U\cap Z) \to \bar U/Z \to \pm H/(Z\cdot V)$$

with $H/(Z\cdot V)$ finite. As $\bar U/Z$ is finitely generated, so is $((1+I)\cap\bar U)/((1+I)\cap\bar U\cap Z)$. But $(1+I)\cap\bar U\cap Z < (1+I)\cap Z < V$, hence $(1+I)\cap\bar U\cap Z$ is finite. Therefore $(1+I)\cap\bar U$ is finitely generated. By Lemmas 4 and 7 it is a nilpotent p-group, hence it is finite.

Let $S \subseteq U$ be an infinite set of unipotent units. As 1 is the only unipotent unit in $\pm H/V \subset \mathbb{Z}[H/V]$ so $\pi(u) = 1$ for all $u \in S$. Hence $\bar\pi\rho(u) = \rho\pi(u) = 1$ and $\rho(u) \in (1+I)\cap\bar U$. It follows that $\rho(u) = \rho(v)$ for some $u, v \in S$. $\square$

For an element $\alpha \in \mathbb{Z}H$ let $C(\alpha)$ denote the centralizer of α in H, i.e., $C(\alpha) = \{h \in H\colon h\alpha = \alpha h\}$.

9. Proposition. *Suppose $\mathbb{Z}G$ contains a nilpotent element $\eta \neq 0$. If there exist a group H with $|H : \mathcal{Z}(H)| < \infty$ and a homomorphism $\phi\colon G \to H$ such that $\phi(\eta) \neq 0$, $|\phi(C(\eta))| = \infty$ and $\phi(\mathcal{U}\mathbb{Z}G)$ is p-nice in $\mathcal{U}\mathbb{Z}H$, then $\mathcal{U}\mathbb{Z}G$ is not finitely generated.*

Proof. We may assume that not all coefficients of η are divisible by p. Set $U = \phi(\mathcal{U}\mathbb{Z}G)$. Let $h_1, h_2, \ldots \in C(\eta)$ map to different elements in H. Consider $X = \{1 + h_i\eta\colon i = 1, 2, \ldots\} \subseteq \mathcal{U}\mathbb{Z}G$. Let overbar denote images under ϕ. If $u = 1 + \bar h_i\bar\eta$, $v = 1 + \bar h_j\bar\eta \in U$, then $u - v = (\bar h_i - \bar h_j)\bar\eta$. Now, for every h_i we are able to choose infinitely many elements $h_{j_1}, h_{j_2}, \ldots$ such that $\operatorname{supp}(\bar h_i\bar\eta) \cap \operatorname{supp}(\bar h_{j_k}\bar\eta) = \varnothing$ and hence $p \not| (\bar h_i - \bar h_{j_k})\bar\eta$. Carrying out the standard diagonal procedure we can find an infinite subset S of X such that $p \not| (u-v)$ for every $u, v \in S$. Our proposition now follows from Theorem 8. $\square$

We now use Theorem 8 to classify finitely generated nilpotent groups G for which $\mathcal{U}\mathbb{Z}G$ is not finitely generated. Recall first the following

10. Theorem (Sehgal [4], p. 176)**.** *Let G be a finitely generated nilpotent group. The group ring $\mathbb{Z}G$ has no nilpotent elements if every finite subgroup of G is normal and either*

(i) *$T(G)$ is abelian, or*

(ii) *$T(G) = A \times E \times K_8$ where E is an elementary abelian 2-group and A is a finite abelian group of odd order m such that the multiplicative order of* $2 \bmod m$ *is odd.* □

Now, using Theorems 8 and 10, we prove the following

11. Theorem. *If G is an infinite nilpotent group and $\mathbb{Z}G$ contains a nonzero nilpotent element, then $\mathcal{U}\mathbb{Z}G$ is not finitely generated.*

Proof. If $\mathcal{U}\mathbb{Z}G$ is generated by $u_1, \dots, u_m$, then G is generated by $\bigcup_{i=1}^m \operatorname{supp}(u_i)$. Therefore we can assume that G is finitely generated. We know that every finitely generated nilpotent group is torsion free-by-finite so we can find a normal torsion free subgroup $H_1 \triangleleft G$ such that $|G : H_1| < \infty$. Lemma 6 assures the existence of a normal in G subgroup $H_2 < H_1$ such that $H_1/H_2 \simeq \mathbb{Z}$. Then the Hirsch rank of G/H_2 is 1.

From Theorem 10 it follows that we are to consider two cases. First let us assume that there exists a finite subgroup $E < G$ which is not normal. As $T(G)$ is the product of its Sylow subgroups we may assume that $E = \langle y \rangle$ where $o(y) = p^n$ for some prime p. We can always find an element $t \in G$ such that $t^{-1}Et \neq E$ and $o(\bar{t}) = \infty$ where $\bar{t}$ is the image of t in G/H_2. This is so because $[T(G), H_2] = 1$ and for any elements $z, x \in G$ such that $o(\bar{z}) = \infty$ and $x^{-1}Ex \neq E$ one of the elements x, xz, z fits. Let $T(G) = S_p \times S_{p'}$ where S_p is the Sylow p-subgroup of $T(G)$.

Now we put: $H = G/(H_2 \cdot S_{p'})$, $\phi\colon G \to H$ is the natural homomorphism, $\eta = (1-y)t(1 + y + \dots + y^{p^n} - 1)$, $V = \phi(S_p) \simeq S_p$, $U = \phi(\mathcal{U}\mathbb{Z}G)$ and check that all assumptions of Proposition 9 are satisfied.

We have the following diagram of natural homomorphisms:

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{Z}[G/T(G)] \\ \downarrow & & \downarrow \\ \mathbb{Z}H = \mathbb{Z}[G/(H_2 \cdot S_{p'})] & \longrightarrow & \mathbb{Z}[G/(H_2 \cdot T(G))] = \mathbb{Z}[H/V] \end{array}$$

and $\mathbb{Z}[G/T(G)]$ has only trivial units because $G/T(G)$ is torsion free nilpotent. Also $H < U$, hence U is p-nice. It is clear that $\phi(\eta) \neq 0$. As $T(G)$ is finite we can find an integer k such that t^k commutes with $T(G)$ and from the construction of η it follows that we have an infinite subset $\{\bar{t}^{kl} \colon l = 1, 2, \dots\} \subseteq \phi(C(\eta))$. Finally, $|H : \mathcal{Z}(H)| < \infty$, because H has Hirsch rank 1. Hence all assumptions of Proposition 9 are satisfied and so $\mathcal{U}\mathbb{Z}G$ is not finitely generated.

Let us now suppose that every finite subgroup of G is normal. As $\mathbb{Z}G$ contains a nonzero nilpotent element we can, by Theorem 10, assume that $T(G) = A \times E \times K_8$, where E is an elementary abelian 2-group and A is a finite abelian group of odd order m such that the multiplicative order of $2 \bmod m$ is even. Recall that we have found torsion free normal subgroups H_1 and H_2 such that G/H_1 is finite and $H_1/H_2 \simeq \mathbb{Z}$.

Let $|A| = m$. If $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then there is at least one $1 \leq i \leq k$ such that the multiplicative order of $2 \bmod p_i^{\alpha_i}$ is even. Indeed, otherwise $2^{l_i} \equiv 1 \pmod{p_i^{\alpha_i}}$ for some odd integers l_i. But then $2^{l_1 l_2 \cdots l_k} \equiv 1 \pmod{m}$ and the multiplicative

order of 2 mod m would be odd. Hence we can choose i so that the multiplicative order of 2 mod $p_i^{\alpha_i}$ is even. Put $p = p_i$ and let S_p be the Sylow p-subgroup of $T(G)$ (i.e., of A). Let $S_{2',p'}$ be the product of all remaining Sylow subgroups of A. Theorem 2 ensures the existence of a nonzero nilpotent element η in $\mathbb{Z}[S_p \times K_8]$.

Now we put: $H = G/(H_2 \cdot S_{2',p'})$, $\phi\colon G \to H$ is the natural homomorphism, $V = \phi(S_p) \simeq S_p$, $U = \phi(\mathcal{U}\mathbb{Z}G)$.

As in the previous case we prove that the assumptions of Proposition 9 are satisfied. This time we have the following diagram of natural homomorphisms:

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{Z}[G/(S_{2',p'} \cdot S_p)] \\ \downarrow & & \downarrow \\ \mathbb{Z}H = \mathbb{Z}[G/(H_2 \cdot S_{2',p'})] & \longrightarrow & \mathbb{Z}[G/(H_2 \cdot S_{2',p'} \cdot S_p)] = \mathbb{Z}[H/V] \end{array}$$

The torsion subgroup of $G/(S_{2',p'} \cdot S_p)$ is $E \times K_8$. It is generally known that there are only trivial units in $\mathbb{Z}[E \times K_8]$. Because we assumed that there are no nontrivial, finite, not normal subgroups in G we may apply Theorem 10 to conclude that there are only trivial units in $\mathbb{Z}[G/(S_{2',p'} \cdot S_p)]$. Also $H < U$ and therefore U is p-nice. Like before we can show that $\phi(\eta) \neq 0$ and $|\phi(C(\eta))| = \infty$. Applying Proposition 9 we obtain that $\mathcal{U}\mathbb{Z}G$ is not finitely generated. $\square$

The main result, Theorem 2, easily follows from the above theorem.

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