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# GENERALIZED POWER MEANS AND INTERPOLATING INEQUALITIES

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Abstract. In this paper, we introduce a multi-parameter family of generalized power means, and use their special properties to provide a new method of interpolating inequalities. We give a different refinement of an inequality of Ky Fan as a particular application of our method.

## 1. Introduction

Interpolation of inequalities extends known inequalities in a systematic way by inserting inequalities between the extremes, or introducing parameters into some functions with special values at those extremes. For example, the $\lambda$-method by D. S. Mitrinović and P. M. Vasić, and the functional equation approach by S. Iwamoto are discussed in [12]. Due to numerous references on inequalities and the large variety of methods, we shall recommend to the interested reader the excellent books [3, 4, 5, 11, 12, 13] and further relevant articles therein.

Let $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbf{R}_+^n$, where $\mathbf{R}_+$ denotes the set of all positive real numbers, and

$$A_n(\mathbf{a}) = \frac{1}{n}\sum_{i=1}^{n} a_i \qquad \text{and} \qquad G_n(\mathbf{a}) = \left[\prod_{i=1}^{n} a_i\right]^{\frac{1}{n}}$$

denote the arithmetic mean and the geometric mean of $a_1, \cdots, a_n$, respectively. One of the most important inequalities, perhaps a keystone of the theory of inequalities, is the arithmetic mean–geometric mean inequality ([3, p. 3]), that is,

$$G_n(\mathbf{a}) \le A_n(\mathbf{a}) \tag{1}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Furthermore, let $H_n(\mathbf{a}) = n[\sum_{i=1}^{n} 1/a_i]^{-1}$ be the harmonic mean of $a_1, \cdots, a_n$. It is also known that ([12])

$$H_n(\mathbf{a}) \le G_n(\mathbf{a}) \le A_n(\mathbf{a}), \tag{2}$$

where, in either case, equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

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A simple example of interpolating inequality (2) is to introduce the so-called symmetric means. For $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbf{R}_+^n$, denote $1/\mathbf{a} = (1/a_1, \cdots, 1/a_n)$, and let

$$\sigma_r(\mathbf{a}) = \sum_{1 \le i_1 < \cdots < i_r \le n} a_{i_1} \cdots a_{i_r},$$

be the $r$th elementary symmetric function of $a_1, \cdots, a_n$; the summation is taken over all possible permutations of $\{i_1, \cdots, i_r\}$, $r = 1, \cdots, n$. Define the $r$th *symmetric mean* of $\mathbf{a}$ as

$$P_n^{[r]}(\mathbf{a}) = \left\{ \begin{bmatrix} n \\ r \end{bmatrix}^{-1} \sigma_r(\mathbf{a}) \right\}^{1/r}, \qquad r = 1, 2, \cdots, n.$$

Moreover we define

$$P_n^{[-r]}(\mathbf{a}) = \left\{ P_n^{[r]}(\frac{1}{\mathbf{a}}) \right\}^{-1}, \qquad r = 1, 2, \cdots, n.$$

It is well-known that ([12, 13]):

$$\begin{aligned} H_n(\mathbf{a}) &= P_n^{[-1]}(\mathbf{a}) \le \cdots \le P_n^{[-n+1]}(\mathbf{a}) \le P_n^{[-n]}(\mathbf{a}) = G_n(\mathbf{a}) \\ &= P_n^{[n]}(\mathbf{a}) \le \cdots \le P_n^{[2]}(\mathbf{a}) \le P_n^{[1]}(\mathbf{a}) = A_n(\mathbf{a}), \end{aligned}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

In the chain of inequalities above, the symmetric means $P_n^{[r]}(\mathbf{a})$, $r = \pm 1, \cdots, \pm n$, can be viewed as a family of $2n - 1$ means with the arithmetic mean $A_n(\mathbf{a})$ as the largest member and the harmonic mean $H_n(\mathbf{a})$ as the smallest. A direct calculation shows that

$$\{G_n(\mathbf{a})\}^n = H_n(\mathbf{a})\{P_n^{[n-1]}(\mathbf{a})\}^{n-1} \quad \text{and} \quad \{G_n(\mathbf{a})\}^n = A_n(\mathbf{a})\{P_n^{[-n+1]}(\mathbf{a})\}^{n-1}.$$

Motivated by different considerations, there has been a great deal of effort devoted to the development of a continuous family of means with diverse applications. Some of the methods are quite elementary but appealing, and some are sophisticated (cf. [9, 18]). For instance, if we define for $x$ real, $x \neq 0$,

$$F(x)(\mathbf{a}) = \left( \frac{a_1^x + a_2^x + \cdots + a_n^x}{n} \right)^{1/x},$$

then it can be verified that ([18]) $G_n(\mathbf{a}) = \lim_{x \longrightarrow 0} F(x)(\mathbf{a})$, $F(x)(\mathbf{a})$ is a monotonic increasing function, and it is continuous everywhere if we define $F(0)(\mathbf{a}) = G_n(\mathbf{a})$. It is clear that

$$F(-1)(\mathbf{a}) = H_n(\mathbf{a}), \qquad F(0)(\mathbf{a}) = G_n(\mathbf{a}), \qquad F(1)(\mathbf{a}) = A_n(\mathbf{a}),$$

and $F(2)(\mathbf{a}) = Q_n(\mathbf{a})$ is the *root mean* of $a_1, \cdots, a_n$. Moreover, denote by $M$ and $m$ the largest element and the smallest element of $a_1, a_2, \cdots, a_n$ respectively. Then from [18] we have

$$\lim_{x \longrightarrow \infty} F(x)(\mathbf{a}) = M \quad \text{and} \quad \lim_{x \longrightarrow -\infty} F(x)(\mathbf{a}) = m.$$

Obviously, $F(x)$ serves as a continuous family of means with particular values as those basic means. Recently, Yang and Wang have introduced different monotonic functions with special values as various means in an endeavour to generalize a famous inequality of Ky Fan ([20]). In the early 1980's, Leach and Sholander studied some multi-variable extended mean values via different functions which possess special properties ([9]).

In the next section, we shall introduce a multi-parameter family of means that will be used to interpolate many useful inequalities including the one of Ky Fan.

## 2. Generalized power means

**Definition 2.1.** Let $\mathbf{a} = (a_1, \cdots, a_n)$, $\mathbf{w} = (w_1, \cdots, w_n) \in \mathbf{R}_+^n$. Set $w_+ = \sum_{i=1}^n w_i$. The $r$th *power mean of* $\mathbf{a}$ *with weight* $\mathbf{w}$, $M_n^{[r]}(\mathbf{a}; \mathbf{w})$ is defined by

$$M_n^{[r]}(\mathbf{a}; \mathbf{w}) = \begin{cases} \left[\prod_{i=1}^n a_i^{w_i}\right]^{\frac{1}{w_+}} & \text{if } r = 0, \\ \left[\frac{\sum_{i=1}^n w_i a_i^r}{w_+}\right]^{\frac{1}{r}} & \text{if } r \neq 0. \end{cases}$$

More generally, define means due to Gini and Bonferroin (cf. [4, p. 189]) $B_n^{r,t}(\mathbf{a}; \mathbf{w})$, $r \geq t \geq 0$, as follows:

$$B_n^{r,t}(\mathbf{a}; \mathbf{w}) = \begin{cases} \left[\prod_{i=1}^n a_i^{w_i a_i^r}\right]^{\frac{1}{\sum_{i=1}^n w_i a_i^r}} & \text{if } r = t, \\ \left[\frac{\sum_{i=1}^n w_i a_i^r}{\sum_{i=1}^n w_i a_i^t}\right]^{\frac{1}{r-t}} & \text{if } r > t. \end{cases}$$

Then $B_n^{r,0}(\mathbf{a}; \mathbf{w}) = M_n^{[r]}(\mathbf{a}; \mathbf{w})$. The *counterharmonic mean* $H_n^{[r]}(\mathbf{a}; \mathbf{w})$, $r \geq 1$, is thereby defined by $H_n^{[r]}(\mathbf{a}; \mathbf{w}) = B_n^{r,r-1}(\mathbf{a}; \mathbf{w})$.

The most important property of the power means is perhaps the following inequality.

**Theorem 2.2** ([4, p. 159])**.** *Let $-\infty < t < r < +\infty$. Then for $\mathbf{a}$, $\mathbf{w} \in \mathbf{R}_+^n$,*

$$M_n^{[t]}(\mathbf{a}; \mathbf{w}) \leq M_n^{[r]}(\mathbf{a}; \mathbf{w}),$$

*with equality if and only if $a_1 = a_2 = \cdots = a_n$.*

*Remark.* When $\mathbf{w} = (1/n, \cdots, 1/n)$, then $M_n^{[1]}(\mathbf{a}; \mathbf{w}) = A_n(\mathbf{a})$, $M_n^{[0]}(\mathbf{a}; \mathbf{w}) = G_n(\mathbf{a})$, and $M_n^{[-1]}(\mathbf{a}; \mathbf{w}) = H_n(\mathbf{a})$. Theorem 2.2 is simply another generalization of the classical inequality (2). With $\mathbf{a}$ and $\mathbf{w}$ fixed, the power mean can be viewed as a monotonic function of $r$.

Now, let $\mathbf{Q}_n \subset \mathbf{R}_+^n$, $(n \geq 2)$, be a non-empty subset, and $\mathbf{f} = (f_1, f_2, \cdots f_m)$, where $f_i : \mathbf{Q}_n \longrightarrow \mathbf{R}_+$, $1 \leq i \leq m$, are distinct functions. Let $w_i > 0$, $1 \leq i \leq m$, and $\triangle(\mathbf{w}) = \triangle(w_1, \cdots, w_m)$ be the $(m-1)$-simplex in $\mathbf{R}^m$ with vertices

$$\mathbf{W}_i = (0, \cdots, 0, 1/w_i, 0, \cdots, 0) \quad \text{where } 1/w_i \text{ is the } i\text{th coordinate}, \quad i = 1, \cdots, m.$$

Thus, if $\mathbf{x} = (x_1, \cdots, x_m) \in \triangle(\mathbf{w})$, then $\sum_{i=1}^m w_i x_i = 1$.

**Definition 2.3.** For $\mathbf{x} \in \triangle(\mathbf{w})$, $\mathbf{a} \in \mathbf{Q}_n$ and $r \geq 0$, define the *generalized power mean* $L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a})$ by

$$L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}) = \begin{cases} \prod_{i=1}^m \{f_i(\mathbf{a})\}^{w_i x_i} & \text{if } r = 0, \\ \left\{\sum_{i=1}^m w_i x_i \left[f_i(\mathbf{a})\right]^r\right\}^{\frac{1}{r}} & \text{if } r > 0. \end{cases}$$

Notice that if we define $f_i(\mathbf{a}) = a_i$ for $\mathbf{a} \in \mathbf{R}_+^n$, and choose $x_i = 1/w_+$, $1 \leq i \leq n$, then

$$L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}) = M_n^{[r]}(\mathbf{a}, \mathbf{w}), \quad r \geq 0.$$

The generalized power means are also well-defined for $r < 0$. For simplicity we will consider only the case when $r \geq 0$. In order to have an inequality comparable to Theorem 2.2, we now introduce an index dominant relation $\succ$ on $\triangle(\mathbf{w})$ as follows.

For $\mathbf{x},\ \mathbf{x}' \in \triangle(\mathbf{w})$, $\mathbf{w} \in \mathbf{R}_+^n$, define $\mathbf{x} \succ \mathbf{x}'$ if $\mathbf{x} = \mathbf{x}'$, or there exists an integer $k$, $1 \le k < m$ such that the following conditions are satisfied:

(3)
$$x_i \ge x_i' \ \text{ for } 1 \le i \le k; \quad x_{k+1} < x_{k+1}'; \ x_i \le x_i' \ \text{ for } k+2 \le i \le m, \quad \text{if } k+2 \le m.$$

Furthermore, for $\mathbf{a} \in \mathbf{Q}_n$, $\mathbf{x} \succ \mathbf{x}'$, and $\mathbf{x} \ne \mathbf{x}'$, we give the definition of the statement $\mathrm{EQ}(\mathbf{f}; \mathbf{x} \succ \mathbf{x}'; \mathbf{a})$ as follows:

$$\mathrm{EQ}(\mathbf{f}; \mathbf{x} \succ \mathbf{x}'; \mathbf{a}): \qquad \sum_{i=1, i\ne k, x_i \ne x_i'}^{m} [f_i(\mathbf{a}) - f_k(\mathbf{a})]^2 = 0 \Longleftrightarrow a_1 = a_2 = \cdots = a_n.$$

For instance, let $\mathbf{f}$ satisfy the following condition: for any pair $(i, j)$, $i \ne j$, $\mathbf{a} \in \mathbf{Q}_n$,

$$f_i(\mathbf{a}) = f_j(\mathbf{a}) \Longleftrightarrow a_1 = a_2 = \cdots = a_n. \tag{4}$$

Then $\mathrm{EQ}(\mathbf{f}; \mathbf{x} \succ \mathbf{x}'; \mathbf{a})$ holds for any $\mathbf{x}, \mathbf{x}'$, $\mathbf{x} \succ \mathbf{x}'$, and $\mathbf{x} \ne \mathbf{x}'$. Some non-trivial examples are given below.

**Example 2.4.** Let $\mathbf{Q}_n = \mathbf{R}_+^n$. The following functions satisfy (4).

(a) $f_i(\mathbf{a}) = P_n^{[r_i]}(\mathbf{a}), \quad 1 \le r_1 < r_2 < \cdots < r_m \le n.$

(b) $f_i(\mathbf{a}) = P_n^{[-r_i]}(\mathbf{a}), \ 1 \le r_1 < r_2 < \cdots < r_m \le n.$

(c) $f_i(\mathbf{a}) = A_n^{[r_i]}(\mathbf{a}), \ 1 \le r_1 < r_2 < \cdots < r_m \le n$, where $A_n^{[r]}(\mathbf{a})$ is defined as

$$A_n^{[r]}(\mathbf{a}) = \{A_n(\mathbf{a})\}^{\frac{n-r}{r(n-1)}} \{G_n(\mathbf{a})\}^{\frac{n(r-1)}{r(n-1)}}.$$

(d) $f_i(\mathbf{a}) = M_n^{[r_i]}(\mathbf{a}; \mathbf{w}), \ 0 \le r_1 < r_2 < \cdots < r_m.$

We now present our main result as follows.

**Theorem 2.5.** (a) *If $0 \le t < r$, $\mathbf{x} \in \triangle(\mathbf{w})$,*

$$L_{n,m}^{[t]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}) \le L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}), \qquad \mathbf{a} \in \mathbf{Q}_n.$$

*Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.*

(b) *Suppose $\mathbf{x} \succ \mathbf{x}'$; $\mathbf{x}, \mathbf{x}' \in \triangle(\mathbf{w})$ and $\mathbf{f}$ satisfies*

$$f_1(\mathbf{a}) \ge f_2(\mathbf{a}) \ge \cdots \ge f_m(\mathbf{a}), \qquad \mathbf{a} \in \mathbf{Q}_n. \tag{5}$$

*Then for any $r \ge 0$,*

$$L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}) \ge L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}'; \mathbf{w}](\mathbf{a}), \qquad \mathbf{a} \in \mathbf{Q}_n.$$

*If $\mathbf{x} \ne \mathbf{x}'$, and $\mathbf{f}$ satisfies the condition $EQ(\mathbf{f}; \mathbf{x} \succ \mathbf{x}'; \mathbf{a})$, then equality holds if and only if $a_1 = a_2 = \cdots = a_n$.*

*Proof.* (a) The inequality follows from Theorem 2.2 because

$$L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}) = M_n^{[r]}\left((f_1(\mathbf{a}), \cdots, f_m(\mathbf{a})), (w_1 x_1, \cdots, w_m x_m)\right).$$

(b) For $\mathbf{a} \in \mathbf{Q}_n$, choose $\beta > 0$ so that $\beta f_m(\mathbf{a}) \ge 1$. First, let us consider the case $r = 0$. For $1 \le i \le k$,

$$(\beta f_i)^{w_i x_i}(\mathbf{a}) = \{\beta f_i(\mathbf{a})\}^{w_i x_i} = (\beta f_i)^{w_i x_i'}(\mathbf{a}) \, (\beta f_i)^{w_i (x_i - x_i')}(\mathbf{a}),$$

hence by (5), we have

$$(\beta f_i)^{w_i x_i}(\mathbf{a}) \ge (\beta f_i)^{w_i x_i'}(\mathbf{a}) \, (\beta f_k)^{w_i (x_i - x_i')}(\mathbf{a}). \tag{6}$$

As $\mathbf{x}, \mathbf{x}' \in \triangle(\mathbf{w})$, $\sum_{i=1}^m w_i x_i = \sum_{i=1}^m w_i x_i' = 1$, hence

$$\sum_{i=1}^k w_i(x_i - x_i') = \sum_{i=k+1}^m w_i(x_i' - x_i). \tag{7}$$

It follows from (5), (6) and (7) that

$$\begin{aligned}
&\beta L_{n,m}^{[r]}[\mathbf{f};\mathbf{x};\mathbf{w}](\mathbf{a}) \\
&\quad= \prod_{i=1}^k (\beta f_i)^{w_i x_i}(\mathbf{a}) \prod_{i=k+1}^m (\beta f_i)^{w_i x_i}(\mathbf{a}) \\
&\quad\geq \prod_{i=1}^k (\beta f_i)^{w_i x_i'}(\mathbf{a}) (\beta f_k)^{\sum_{i=1}^k w_i(x_i - x_i')}(\mathbf{a}) \prod_{i=k+1}^m (\beta f_i)^{w_i x_i}(\mathbf{a}) \\
&\quad= \prod_{i=1}^k (\beta f_i)^{w_i x_i'}(\mathbf{a}) (\beta f_k)^{\sum_{i=k+1}^m w_i(x_i' - x_i)}(\mathbf{a}) \prod_{i=k+1}^m (\beta f_i)^{w_i x_i}(\mathbf{a}) \\
&\quad\geq \prod_{i=1}^k (\beta f_i)^{w_i x_i'}(\mathbf{a}) \prod_{i=k+1}^m (\beta f_i)^{w_i(x_i' - x_i)}(\mathbf{a}) \prod_{i=k+1}^m (\beta f_i)^{w_i x_i}(\mathbf{a}) \\
&\quad= \beta \prod_{i=1}^m f_i^{w_i x_i'}(\mathbf{a}) \\
&\quad= \beta L_{n,m}^{[r]}[\mathbf{f};\mathbf{x}';\mathbf{w}](\mathbf{a}).
\end{aligned}$$

Equality holds if and only if for $i \neq k$, $1 \leq i \leq m$, we have

$$(\beta f_i)^{w_i(x_i - x_i')}(\mathbf{a}) = (\beta f_k)^{w_i(x_i - x_i')}(\mathbf{a}). \tag{8}$$

Since $\mathbf{x} \succ \mathbf{x}'$, there exists at least one $i$ such that $x_i \neq x_i'$, hence (8) implies that $f_i(\mathbf{a}) = f_k(\mathbf{a})$ if $x_i \neq x_i'$. By the equality condition $\mathrm{EQ}(\mathbf{f};\mathbf{x} \succ \mathbf{x}';\mathbf{a})$ we have $a_1 = a_2 = \cdots = a_n$.

For $r > 0$, the proof is similar.

$$\begin{aligned}
&\beta^r \left\{ L_{n,m}^{[r]}[\mathbf{f};\mathbf{x};\mathbf{w}](\mathbf{a}) \right\}^r \\
&\quad\geq \sum_{i=1}^k w_i x_i'(\beta f_i)^r(\mathbf{a}) + \sum_{i=1}^k w_i(x_i - x_i')(\beta f_k)^r(\mathbf{a}) + \sum_{i=k+1}^m w_i x_i(\beta f_i)^r(\mathbf{a}) \\
&\quad\geq \sum_{i=1}^k w_i x_i'(\beta f_i)^r(\mathbf{a}) + \sum_{i=k+1}^m w_i(x_i' - x_i)(\beta f_k)^r(\mathbf{a}) + \sum_{i=k+1}^m w_i x_i(\beta f_i)^r(\mathbf{a}) \\
&\quad\geq \sum_{i=1}^k w_i x_i'(\beta f_i)^r(\mathbf{a}) + \sum_{i=k+1}^m w_i(x_i' - x_i)(\beta f_i)^r(\mathbf{a}) + \sum_{i=k+1}^m w_i x_i(\beta f_i)^r(\mathbf{a}) \\
&\quad= \beta^r \sum_{i=1}^m w_i x_i' f_i^r(\mathbf{a}) \\
&\quad= \beta^r \left\{ L_{n,m}^{[r]}[\mathbf{f};\mathbf{x}';\mathbf{w}](\mathbf{a}) \right\}^r.
\end{aligned}$$

Equality holds if and only if

$$w_i(x_i - x_i')f_i^r(\mathbf{a}) = w_i(x_i - x_i')f_k^r(\mathbf{a}), \qquad i \neq k, \qquad 1 \leq i \leq m. \tag{9}$$

Again, we use the hypothesis $\mathrm{EQ}(\mathbf{f}; \mathbf{x} \succ \mathbf{x}'; \mathbf{a})$ concluding that $a_1 = a_2 = \cdots = a_n$. $\square$

*Remarks* 2.6. (i) Theorem 2.5 (a) becomes Theorem 2.2 if we choose $f_i(\mathbf{a}) = a_i$ for $i = 1, \cdots, n$. (ii) We can apply Theorem 2.5 (b) to the functions in Example 2.4. The following special case of Theorem 2.5 (b) was proved in [7]: $m = 3$, $w_1 = 1$, $w_2 = n-1$, $w_3 = n/2$, $n \geq 3$, $f_1(\mathbf{a}) = A_n(\mathbf{a})$, $f_2(\mathbf{a}) = \{[A_n(\mathbf{a})]^{n-2}[G_n(\mathbf{a})]^n\}^{\frac{1}{2(n-1)}}$, and $f_3(\mathbf{a}) = G_n(\mathbf{a})$.

Theorem 2.5 (b) generalizes many important inequalities (more applications will be given in the next section). For instance, let us consider some inequalities involving $H_n^{[r]}(\mathbf{a}; \mathbf{w})$, or $B_n^{r,t}(\mathbf{a}; \mathbf{w})$ below. For $\mathbf{a} \in \mathbf{R}_+^n$, we can assume that $a_1 \geq a_2 \geq \cdots \geq a_n$ (by rearranging the subscripts). Choose $\beta > 0$ so that $\beta a_n \geq 1$. Let $r > t \geq 0$. For $\mathbf{w} \in \mathbf{R}_+^n$, set

$$u = \sum_{i=1}^n w_i a_i^r, \qquad v = \sum_{i=1}^n w_i a_i^t,$$

$$x_j = a_j^r/u, \quad \text{and} \quad x_j' = a_j^t/v, \quad 1 \leq j \leq n. \tag{10}$$

Notice that

$$x_j = \frac{(\beta a_j)^r}{\sum_{i=1}^n w_i(\beta a_i)^r} \quad \text{and} \quad x_j' = \frac{(\beta a_j)^t}{\sum_{i=1}^n w_i(\beta a_i)^t}.$$

Set $c = \sum_{i=1}^n w_i(\beta a_i)^r / \sum_{i=1}^n w_i(\beta a_i)^t$, as $r > t$ and $\beta a_n \geq 1$, we have $c \geq 1$.

**Lemma 2.7.** *Let* $\mathbf{x}$, $\mathbf{x}' \in \mathbf{R}_+^n$ *be defined by* (10). *Then* $\mathbf{x}$, $\mathbf{x}' \in \triangle(\mathbf{w})$ *and* $\mathbf{x} \succ \mathbf{x}'$.

*Proof.* It is clear that

$$\sum_{i=1}^n w_i x_i = \sum_{i=1}^n w_i x_i' = 1. \tag{11}$$

Hence, $\mathbf{x}$, $\mathbf{x}' \in \triangle(\mathbf{w})$. If $a_1 = \cdots = a_n$, then $\mathbf{x} = \mathbf{x}'$. Suppose now that not all $a_j$'s are equal. Since $x_j'/x_j = c(\beta a_j)^{t-r}$, we obtain

$$x_j' \leq x_j \ (\text{resp. } x_j' \geq x_j) \iff c \leq (\beta a_j)^{r-t} \ (\text{resp. } c \geq (\beta a_j)^{r-t}). \tag{12}$$

If $x_j' \leq x_j$ (resp. $x_j' \geq x_j$) for $1 \leq j \leq n$, then we shall have $a_1 = a_2 = \cdots = a_n$, and this is a contradiction. Clearly if $x_j' \leq x_j$ for $1 \leq j \leq n$, then $x_j' = x_j$ for $1 \leq j \leq n$ by (11), and so, $c = (\beta a_j)^{r-t}$, $1 \leq j \leq n$ by (12). Thus, there exists a largest positive integer $k$, $k < n$, so that either

$$x_k' \leq x_k \quad \text{and} \quad x_{k+1}' > x_{k+1}, \quad \text{or} \tag{13}$$

$$x_k' \geq x_k \quad \text{and} \quad x_{k+1}' < x_{k+1}. \tag{14}$$

Suppose (14) holds, then by (12)

$$(\beta a_k)^{r-t} \leq c \quad \text{and} \quad c < (\beta a_{k+1})^{r-t}.$$

Since $a_k \geq a_{k+1}$ and $r > t$, this is impossible. Hence (13) holds. If $n \geq k+2$, we need to verify that $x_j' \geq x_j$ for $k+2 \leq j \leq n$. If not, there exists $s \geq k+2$ such that $x_s > x_s'$. Since $x_{k+1}' > x_{k+1}$ by (13), again by (12) we have

$$c < (\beta a_s)^{r-t} \quad \text{and} \quad c < (\beta a_{k+1})^{r-t},$$

which will imply that $a_s > a_{k+1}$, a contradiction with the arrangement of the $a_j$'s. This completes the proof. □

**Theorem 2.8** (cf. [4]). *Let* $\mathbf{a},\ \mathbf{w} \in \mathbf{R}^n_+$.

(a) *Suppose* $r_1 > t_1$, $r_2 > t_2 \geq t_1 \geq 0$, *and* $t_2 - t_1 \leq r_2 - r_1$. *Then*

$$B_n^{r_1,t_1}(\mathbf{a};\mathbf{w}) \leq B_n^{r_2,t_2}(\mathbf{a};\mathbf{w}).$$

(b) *For* $1 \leq t < r$,

$$H_n^{[t]}(\mathbf{a};\mathbf{w}) \leq H_n^{[r]}(\mathbf{a};\mathbf{w}).$$

(c) *For* $1 < r$,

$$M_n^{[r]}(\mathbf{a};\mathbf{w}) \leq H_n^{[r]}(\mathbf{a};\mathbf{w}).$$

*Equality holds in* (a) (*resp.* (b), (c)) *if and only if* $a_1 = a_2 = \cdots = a_n$.

*Proof.* For $\mathbf{a} \in \mathbf{R}^n_+$, we may assume that $a_1 \geq a_2 \geq \cdots \geq a_n$ (by rearranging the subscripts if necessary). Set $f_i(\mathbf{a}) = a_i$, $1 \leq i \leq n$.

(a) Define $x'_j = a_j^{t_1} / \sum_{i=1}^n w_i a_i^{t_1}$, and $x_j = a_j^{r_1} / \sum_{i=1}^n w_i a_i^{r_1}$, $1 \leq j \leq n$, then $\mathbf{x} \succ \mathbf{x}'$ by Lemma 2.7. Hence by Theorem 2.5

$$\begin{aligned} B_n^{r_1,t_1}(\mathbf{a};\mathbf{w}) &= L_{n,n}^{[r_1-t_1]}[\mathbf{f};\mathbf{x}';\mathbf{w}](\mathbf{a}) \\ &\leq L_{n,n}^{[r_1-t_1]}[\mathbf{f};\mathbf{x};\mathbf{w}](\mathbf{a}) \\ &\leq L_{n,n}^{[r_2-t_2]}[\mathbf{f};\mathbf{x};\mathbf{w}](\mathbf{a}) \\ &= B_n^{r_2,t_2}(\mathbf{a};\mathbf{w}). \end{aligned}$$

(b) Set $x'_j = a_j^{t-1} / \sum_{i=1}^n w_i a_i^{t-1}$ and $x_j = a_j^{r-1} / \sum_{i=1}^n w_i a_i^{r-1}$, $1 \leq i \leq n$, then $\mathbf{x} \succ \mathbf{x}'$ by Lemma 2.7 and

$$H_n^{[t]}(\mathbf{a};\mathbf{w}) = L_{n,n}^{[1]}[\mathbf{f};\mathbf{x}';\mathbf{w}](\mathbf{a}) \leq L_{n,n}^{[1]}[\mathbf{f};\mathbf{x};\mathbf{w}](\mathbf{a}) = H_n^{[r]}(\mathbf{a};\mathbf{w}).$$

Since $H_n^{[t]}(\mathbf{a};\mathbf{w}) = B_n^{t,t-1}(\mathbf{a};\mathbf{w})$, this inequality is also a corollary to (a).

(c) Take $x'_j = 1/w_+$, and $x_j = a_j / \sum_{i=1}^n w_i a_i^r$, $1 \leq j \leq n$. Clearly $\mathbf{x} \succ \mathbf{x}'$ and so

$$M_n^{[r]}(\mathbf{a};\mathbf{w}) = L_{n,n}^{[r]}[\mathbf{f};\mathbf{x}';\mathbf{w}](\mathbf{a}) \leq L_{n,n}^{[r]}[\mathbf{f};\mathbf{x};\mathbf{w}](\mathbf{a}) = H_n^{[r]}(\mathbf{a};\mathbf{w}).$$

For equality, since the proofs are similar, we shall only give the proof of (a). We need to show that the condition $\mathrm{EQ}(\mathbf{f};\mathbf{x} \succ \mathbf{x}';\mathbf{a})$ is satisfied. To confirm this, it suffices to verify that if the $a_j$'s are not all equal, then there exists an $i$, $i \neq k$, so that $f_i(\mathbf{a}) \neq f_k(\mathbf{a})$, where $x'_{k+1} > x_{k+1}$ since (12) and (13) still hold. Hence by (12) $c > (\beta a_{k+1})^{r_1-t_1}$. We shall complete the proof by verifying that $f_{k+1}(\mathbf{a}) \neq f_k(\mathbf{a})$, otherwise, $a_k = f_k(\mathbf{a}) = f_{k+1}(\mathbf{a}) = a_{k+1}$. Hence, $c > (\beta a_k)^{r_1-t_1}$, again, by (12), $x'_k > x_k$ which contradicts (13). □

## 3. Interpolation of inequalities

In this section, we shall apply Theorem 2.5 (b) to refine many well-known inequalities including an inequality of Ky Fan ([12]). We adopt the same notations introduced in sections 1 and 2 unless otherwise specified. For simplicity of the statements and the intuition of the arguments, we will focus on the interpolation of double inequalities with three quantities. The general case of interpolating a chain of any finite number of inequalities is similar.

As a special case of Definition 2.3, let $m = 3$, $\triangle(\mathbf{w})$ be the triangle in $\mathbf{R}^3$ with any fixed $\mathbf{w} \in \mathbf{R}^3_+$, and

$$f_1(\mathbf{a}) = A_n(\mathbf{a}), \qquad f_2(\mathbf{a}) = G_n(\mathbf{a}), \qquad f_3(\mathbf{a}) = H_n(\mathbf{a}).$$

For a fixed $r \geq 0$, define a 3-variable function $\Phi(\mathbf{x})$ of $\mathbf{x} = (x_1, x_2, x_3)$ on $\triangle(\mathbf{w})$ via Definition 2.3 as

$$\Phi(\mathbf{x})(\mathbf{a}) = L^{[r]}_{n,3}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}).$$

Then it is clear that,

$$\Phi(\mathbf{W}_1)(\mathbf{a}) = A_n(\mathbf{a}), \qquad \Phi(\mathbf{W}_2)(\mathbf{a}) = G_n(\mathbf{a}), \qquad \Phi(\mathbf{W}_3)(\mathbf{a}) = H_n(\mathbf{a}).$$

From Theorem 2.5 (b), we have

**Theorem 3.1.** *If* $\mathbf{x},\ \mathbf{x}' \in \triangle(\mathbf{w})$, $\mathbf{w} \in \mathbf{R}^n_+$, *and* $\mathbf{x} \succ \mathbf{x}'$, $\mathbf{x} \neq \mathbf{x}'$, *then*

$$\Phi(\mathbf{x})(\mathbf{a}) \geq \Phi(\mathbf{x}')(\mathbf{a})$$

*with equality holds if and only if* $a_1 = a_2 = \cdots = a_n$.

For each $\mathbf{w} \in \mathbf{R}^3_+$, since $\sum_{i=1}^3 w_i x_i = 1$, $\Phi(\mathbf{x})$ provides a two-parameter continuous family of means for $a_1, a_2, \cdots, a_n$ with $A_n(\mathbf{a})$ as the largest member and $H_n(\mathbf{a})$ as the smallest.

By applying the same argument as we used in Theorem 3.1, we are able to refine the following well-known inequality of Ky Fan which has been the subject of a number of articles ([1, 2, 3, 10, 19, 20]):

$$\frac{H_n(\mathbf{a})}{H'_n(\mathbf{a})} \leq \frac{G_n(\mathbf{a})}{G'_n(\mathbf{a})} \leq \frac{A_n(\mathbf{a})}{A'_n(\mathbf{a})}, \tag{15}$$

where $A'_n(\mathbf{a})$, $G'_n(\mathbf{a})$ and $H'_n(\mathbf{a})$ are the arithmetic mean, geometric mean, and harmonic mean of $1-a_1, \cdots, 1-a_n$, respectively, when $a_i \in (0, 1/2]$, $i = 1, 2, \cdots, n$.

Recently, Yang and Wang have introduced two continuous monotonic functions $p(t)(\mathbf{a})$ and $q(t)(\mathbf{a})$ such that ([20])

$$\frac{H_n(\mathbf{a})}{H'_n(\mathbf{a})} \leq p(t)(\mathbf{a}) \leq \frac{G_n(\mathbf{a})}{G'_n(\mathbf{a})} \leq q(t)(\mathbf{a}) \leq \frac{A_n(\mathbf{a})}{A'_n(\mathbf{a})} \tag{16}$$

with $p(1/n) = H_n(\mathbf{a})/H'_n(\mathbf{a})$, $p(0) = G_n(\mathbf{a})/G'_n(\mathbf{a}) = q(0)$, and $q(1/n) = A_n(\mathbf{a})/A'_n(\mathbf{a})$.

Now, assume that $\triangle(\mathbf{w})$ is the same as in Theorem 3.1. Let

$$f_1(\mathbf{a}) = \frac{A_n(\mathbf{a})}{A'_n(\mathbf{a})}, \qquad f_2(\mathbf{a}) = \frac{G_n(\mathbf{a})}{G'_n(\mathbf{a})}, \qquad f_3(\mathbf{a}) = \frac{H_n(\mathbf{a})}{H'_n(\mathbf{a})},$$

and for a fixed $r \geq 0$, let

$$\Psi(\mathbf{x})(\mathbf{a}) = L^{[r]}_{n,3}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}).$$

Then we have

$$\Psi(\mathbf{W}_1)(\mathbf{a}) = \frac{A_n(\mathbf{a})}{A'_n(\mathbf{a})}, \qquad \Psi(\mathbf{W}_2)(\mathbf{a}) = \frac{G_n(\mathbf{a})}{G'_n(\mathbf{a})}, \qquad \text{and} \qquad \Psi(\mathbf{W}_3)(\mathbf{a}) = \frac{H_n(\mathbf{a})}{H'_n(\mathbf{a})}.$$

We obtain the following counterpart of Theorem 3.1 which is simply another refinement of Fan's inequality in terms of a two-parameter family of inequalities.

**Theorem 3.1′.** *For* $\mathbf{w} \in \mathbf{R}^3_+$, $\mathbf{x},\ \mathbf{x}' \in \triangle(\mathbf{w})$, $\mathbf{x} \succ \mathbf{x}'$ *and* $\mathbf{x} \neq \mathbf{x}'$, *then*

$$\Psi(\mathbf{x})(\mathbf{a}) \geq \Psi(\mathbf{x}')(\mathbf{a})$$

*with equality if and only if* $a_1 = \cdots = a_n$.

## 4. Concluding remarks

The original inequality of Ky Fan was

$$\frac{G_n(\mathbf{a})}{G'_n(\mathbf{a})} \le \frac{A_n(\mathbf{a})}{A'_n(\mathbf{a})}.$$

Later on, many articles developed it into the form of (15) with various techniques. As a matter of fact, since there are many symmetric means $P_n^{[r]}(\mathbf{a})$, $r = 1, \cdots, n$, lying between $A_n(\mathbf{a})$ and $G_n(\mathbf{a})$, and $P_n^{[-r]}(\mathbf{a})$, $r = 1, \cdots, n$, lying between $G_n(\mathbf{a})$ and $H_n(\mathbf{a})$, the following more general inequalities of Fan-type are also valid [12, p. 32]:

$$\begin{aligned}\frac{H_n(\mathbf{a})}{H'_n(\mathbf{a})} &= \frac{P_n^{[-1]}(\mathbf{a})}{P_n^{[-1]\prime}(\mathbf{a})} \le \frac{P_n^{[-2]}(\mathbf{a})}{P_n^{[-2]\prime}(\mathbf{a})} \le \cdots \le \frac{P_n^{[-n+1]}(\mathbf{a})}{P_n^{[-n+1]\prime}(\mathbf{a})} \le \frac{P_n^{[-n]}(\mathbf{a})}{P_n^{[-n]\prime}(\mathbf{a})} = \frac{G_n(\mathbf{a})}{G'_n(\mathbf{a})} \\ &= \frac{P_n^{[n]}(\mathbf{a})}{P_n^{[n]\prime}(\mathbf{a})} \le \frac{P_n^{[n-1]}(\mathbf{a})}{P_n^{[n-1]\prime}(\mathbf{a})} \le \cdots \le \frac{P_n^{[2]}(\mathbf{a})}{P_n^{[2]\prime}(\mathbf{a})} \le \frac{P_n^{[1]}(\mathbf{a})}{P_n^{[1]\prime}(\mathbf{a})} = \frac{A_n(\mathbf{a})}{A'_n(\mathbf{a})},\end{aligned}$$

where every quantity with a prime in the denominator represents the corresponding mean of $1 - a_1, \cdots, 1 - a_n$, for $a_i \in (0, 1/2]$, $i = 1, 2, \cdots, n$.

We may use a $2(n-1)$-simplex $\triangle(\mathbf{w})$ in $\mathbf{R}^{2n-1}$ as described in section 2, in the same spirit of Theorems 3.1, 3.1′ to generalize the chain of inequalities of Fan-type above, or the chain of symmetric mean inequalities in section 1 to a $(2n-2)$-parameter family of inequalities. More specifically, Let $\triangle(\mathbf{w})$ be the $2(n-1)$-simplex in $\mathbf{R}^{2n-1}$ with vertices $\mathbf{W}_i$ as defined in section 2, where $\mathbf{w} \in \mathbf{R}_+^{2n-1}$. Set

$$f_i(\mathbf{a}) = P_n^{[i]}(\mathbf{a}),\ 1 \le i \le n;\ \text{and}\ \ f_{i+n}(\mathbf{a}) = P_n^{[-i]}(\mathbf{a}),\ 1 \le i \le n-1.$$

Then we can view $\triangle(\mathbf{w})$ as an index domain for $\Phi(\mathbf{x})$ whose special values on the vertices are symmetric means, and for $\Psi(\mathbf{x})$ whose special values on the vertices are the quotients of symmetric means in the inequality of Fan-type. In contrast with [9], we may call the functions $\Phi(\mathbf{x})(\mathbf{a})$ and $\Psi(\mathbf{x})(\mathbf{a})$ multi-variable extended mean values of $a_1, \cdots, a_n$ and multi-variable extended mean value quotients of Fan-type. In [9], Leach and Sholander demonstrated some very interesting properties of multi-variable extended mean values. Likewise, one could carry out a similar discussion for our generalized power means.

Means and their inequalities have always been an important resource for geometric extremum problems. In his wonderful book *Induction and Analogy in Mathematics* [16], Pólya pointed out the remarkable analogy between the arithmetic mean-geometric mean inequality and the classical isoperimetric inequality for a simple closed plane curve. Therefore, any refinement of one object will likely imply a similar refinement of the other. In Ivan Niven's book *Maxima and Minima Without Calculus* [14] that has enlightened many interesting articles in MAA journals, inequalities involving different means are key ingredients. In fact, geometric inequalities which are derived from inequalities of means are collected in many books such as [6, 13, 14, 16, 17] etc. Regarding $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ as the side lengths of an $n$-sided plane polygon $\mathcal{P}_n$, in [7], we have used some refined inequalities of means to establish many geometric inequalities of isoperimetric type for polygons. In [13], one can find many geometric inequalities for an $n$-simplex $\mathcal{S}_n$ in $\mathbf{R}^m$ where $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ represents a collection of the areas of its $n$ faces, or a collection of its $n$ altitudes, and so on, as immediate consequences of some inequalities in

means. The method of interpolating inequalities is certainly applicable to refining geometric inequalities. We have used the results of this paper and obtained some new interesting geometric inequalities in [8].

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