

## A UNIVERSAL FUNCTIONAL EQUATION

CARSTEN ELSNER

(Communicated by Hal L. Smith)

ABSTRACT. It is shown that the  $S$ -chains solving Rubel's universal fourth-order differential equation also satisfy a third-order functional equation.

### 1. INTRODUCTION AND STATEMENT OF RESULTS

In 1981, L. A. Rubel [3] published the following result concerning the approximation of continuous functions by solutions of algebraic differential equations (ADE):

*There exists a nontrivial fourth-order ADE such that any real continuous function on the real axis can be uniformly approximated by the  $C^\infty$ -solutions of this ADE. One such specific ADE is*

$$(1.1) \quad 3y'^4 y'' y''''^2 - 4y'^4 y''''^2 y'''' + 6y'^3 y''^2 y'''' y'''' + 24y'^2 y''^4 y'''' - 12y'^3 y'' y''''^3 - 29y'^2 y''^3 y''''^2 + 12y''^7 = 0.$$

The basic idea in proving this theorem is to investigate the function

$$(1.2) \quad y = A \cdot f(\alpha t + \beta) + B \quad (-1 \leq \alpha t + \beta \leq 1),$$

where  $A, B, \alpha$  and  $\beta$  are real parameters, and where the function  $f$  is given by

$$(1.3) \quad f(s) := \int_{-1}^s e^{-1/(1-u^2)} du \quad (-1 \leq s \leq 1).$$

The graph of  $f$  is called a *primitive  $S$ -module*. A survey on the theory of universal formulae and universal differential equations can be found in [1]. It seems to be obvious that the degree of the ADE given in (1.1) is best-possible. By using different  $S$ -modules constructed with

$$\int_{-\pi/2}^s e^{-1/\cos u} du \quad \left(-\frac{\pi}{2} \leq s \leq \frac{\pi}{2}\right),$$

the author [2] gets a third-order ADE with coefficients depending on a certain  $\varepsilon$ -neighborhood and on the Lipschitz-class of the continuous functions.

**Definition 1.1.** Let  $D$  be any set of real numbers, and let  $P(u_1, \dots, u_m)$  be a polynomial in  $u_1, \dots, u_m$  with integer coefficients, which does not vanish identically. A real function  $y = y(x)$  with domain  $D$  is said to be a local solution of a functional

---

Received by the editors May 2, 1997.

1991 *Mathematics Subject Classification*. Primary 34K05, 34A34.

*Key words and phrases*. Functional-differential equations, nonlinear ordinary differential equations.

equation of degree  $n \geq 0$  (with respect to the polynomial  $P$ ), if for every point  $x_0$  from  $D$  there are distinct rationals  $r_1, \dots, r_k$  and some  $\varepsilon > 0$ , such that

$$P(y(x+r_1), \dots, y(x+r_k), y'(x+r_1), \dots, y'(x+r_k), \dots, y^{(n)}(x+r_k)) = 0$$

holds for every  $x$  with  $x_0 \leq x \leq x_0 + \varepsilon$  and  $x + r_\kappa \in D$  ( $1 \leq \kappa \leq k$ ).

Now we shall show that the functions from (1.2) are local solutions of a functional equation of degree 3, and that the corresponding polynomial  $P$  can be computed explicitly.

**Theorem 1.1.** *Every real continuous function  $F$  on the real axis can be uniformly approximated by  $C^\infty$ -functions, which are all local solutions of a functional equation of degree 3. The corresponding polynomial  $P(u_1, \dots, u_9)$  has 206 terms, its greatest coefficient is 358 318 080 000, and it does not depend on  $F$ .*

The polynomial  $P$  from this theorem can be simplified in a certain sense. Put

$$(1.4) \quad u := \frac{y_1' y_1'''}{y_1''^2}, \quad v := \frac{y_2' y_2'''}{y_2''^2}, \quad w := \frac{y_3' y_3'''}{y_3''^2},$$

where

$$(1.5) \quad y_i := y \left( x + \frac{i-1}{n} \right) \quad (i = 1, 2, 3; n \in \mathbb{Z}_{>0}).$$

Then  $P$  can be written as a polynomial in  $u$ ,  $v$  and  $w$  of degree 8 with respect to each variable.

## 2. ON THEOREM 1.1: EXISTENCE PROOF OF A FUNCTIONAL EQUATION OF DEGREE 3 AND THEORETICAL FOUNDATION OF ITS COMPUTATION

The approximating function  $Y$  of any continuous function  $F$  is given by ‘‘S-chains’’, which means a  $C^\infty$ -function that consists of S-modules (1.2) pieced together. L. A. Rubel [3] has proved that  $F$  is approximated uniformly by certain S-chains  $Y$  of this type. So it remains to prove the existence of a functional equation, having every S-chain as a local solution.

Let  $x_0$  be any real number. Obviously, for some sufficiently large positive integer  $n$  every real number  $x$  with  $x_0 \leq x \leq x_0 + 1/n$  belongs to the domain of some function

$$(2.1) \quad y = A \cdot f(\alpha x + \beta) + B \quad (-1 \leq \alpha x + \beta \leq 1)$$

from (1.2) and (1.3), which represents a certain section from the approximating function  $Y$ . Let  $\alpha \geq 0$  first. Without loss of generality, we may assume that

$$-1 \leq \alpha x_0 + \beta < 1;$$

hence, for sufficiently large  $n$ , we have

$$\alpha x_0 + \beta < 1 - \frac{3\alpha}{n}.$$

By  $x_0 \leq x \leq x_0 + 1/n$ , it follows that

$$-1 \leq \alpha x + \beta < 1 - \frac{2\alpha}{n},$$

or

$$-1 + \frac{2\alpha}{n} \leq \alpha x + \beta + \frac{2\alpha}{n} < 1.$$

For brevity, we define

$$s_i := \alpha x + \beta + \frac{\alpha(i-1)}{n} \quad (i = 1, 2, 3)$$

for any  $x$  with  $x_0 \leq x \leq x_0 + 1/n$ , which means that  $-1 \leq s_i < 1$  for  $i = 1, 2, 3$ . Thus, passing from  $x$  to  $x + 1/n$  and to  $x + 2/n$ , we do not leave the domain of the function  $y$  in (2.1). If  $\alpha < 0$ , we may assume that  $1 \geq \alpha x_0 + \beta > -1$  and proceed in the same way. From

$$1 \geq \alpha x + \beta > -1 - \frac{2\alpha}{n}$$

we conclude that  $1 \geq s_i > -1$  ( $i = 1, 2, 3$ ).

First we treat the case where  $y_1'' y_2'' y_3''(x) \neq 0$  with  $y_i$  from (1.5). Differentiating  $f$  with respect to  $s$ , we obtain from (1.3) that

$$(2.2) \quad \begin{aligned} f'(s) &= e^{-1/(1-s^2)}, & f''(s) &= \frac{-2s}{(1-s^2)^2} e^{-1/(1-s^2)}, \\ f'''(s) &= \frac{6s^4 - 2}{(1-s^2)^4} e^{-1/(1-s^2)}. \end{aligned}$$

Note that  $f'(\pm 1) = 0$ ,  $f''(\pm 1) = 0$  and  $f'''(\pm 1) = 0$  do exist. Differentiating the functions  $y_i$  from (1.5) with respect to  $x$ , we have, using (2.1),

$$(2.3) \quad y_i' = \alpha A f'(s_i), \quad y_i'' = \alpha^2 A f''(s_i), \quad y_i''' = \alpha^3 A f'''(s_i) \quad (i = 1, 2, 3).$$

To make the desired functional equation independent from special values of  $A$ ,  $B$ ,  $\alpha$  and  $\beta$ , we have to eliminate these parameters. Thus, we get for every  $i = 1, 2, 3$ :

$$\begin{aligned} y_i' y_i''' &= \alpha^4 A^2 f'(s_i) f'''(s_i) = \frac{(6s_i^4 - 2)\alpha^4 A^2}{(1-s_i^2)^4} e^{-2/(1-s_i^2)} \\ &= \frac{3s_i^4 - 1}{2s_i^2} y_i''^2 \quad (\text{by (2.2)}). \end{aligned}$$

From (1.4), we conclude for  $y_i'' \neq 0$ :

$$u = \frac{3s_1^4 - 1}{2s_1^2}, \quad v = \frac{3s_2^4 - 1}{2s_2^2}, \quad w = \frac{3s_3^4 - 1}{2s_3^2}.$$

In what follows, we work with  $w$ ; there are no essential differences in using  $u$  or  $v$ . Since  $s_3^2 \geq 0$ , we can solve the equation for  $s_3^2$ :

$$s_3^2 = \frac{w}{3} + \sqrt{\left(\frac{w}{3}\right)^2 + \frac{1}{3}}.$$

Let  $\sigma_3$  denote the sign of  $s_3$ ; hence

$$s_3 = \sigma_3 \sqrt{\frac{w}{3} + \sqrt{\left(\frac{w}{3}\right)^2 + \frac{1}{3}}}.$$

Note that  $\sigma_3 \in \{-1, 1\}$  depends on  $x$ . It is  $s_3 \neq 0$ , since otherwise  $f''(s_3)$  and  $y_3''$  vanish by (2.2) and (2.3). But this contradicts our assumption on  $y_3''$ . We also have

$$s_2 = \sigma_2 \sqrt{\frac{v}{3} + \sqrt{\left(\frac{v}{3}\right)^2 + \frac{1}{3}}} \quad (\sigma_2 = \text{sign } s_2 \in \{-1, 1\})$$

and

$$s_1 = \sigma_1 \sqrt{\frac{u}{3} + \sqrt{\left(\frac{u}{3}\right)^2 + \frac{1}{3}}} \quad (\sigma_1 = \text{sign } s_1 \in \{-1, 1\}).$$

There is a simple relation between  $s_1, s_2$  and  $s_3$ , namely

$$2s_2 = 2\alpha x + 2\beta + \frac{2\alpha}{n} = s_1 + s_3,$$

or

$$2s_1 s_3 = 4s_2^2 - s_1^2 - s_3^2.$$

Squaring this equation a second time, we get

$$4s_1^2 s_3^2 = (4s_2^2 - s_1^2 - s_3^2)^2.$$

Putting in the expressions for  $s_1, s_2$  and  $s_3$  and multiplying by 9, we get an algebraic relation between  $u, v$  and  $w$ :

$$(2.4) \quad 4 \cdot \left(u + \sqrt{u^2 + 3}\right) \left(w + \sqrt{w^2 + 3}\right) \\ = \left\{4 \left(v + \sqrt{v^2 + 3}\right) - u - \sqrt{u^2 + 3} - w - \sqrt{w^2 + 3}\right\}^2.$$

This equation and (1.4) prove that every S-chain is a local solution of a certain functional equation of degree 3. Note that the cases where  $y_1'' y_2'' y_3''(x) = 0$  do not belong to any essential exception, since one may argue with the continuity of all occurring functions. Therefore the functional equation holds for such  $x$ , too.

Similar to the proof of Rubel's universal differential equation, a surprising amount of cancellation takes place if we eliminate the square roots in (2.4) step by step. First we introduce some useful notations:

$$a := u^2 + 3, \quad b := v^2 + 3, \quad c := w^2 + 3; \\ A := 4v - u + w, \quad B := -4(4v - u - w), \quad C := 4v + u - w, \\ D := uw + 4uv + 4vw - u^2 - 16v^2 - w^2 - 27.$$

Hence, (2.4) takes the form

$$A\sqrt{a} + B\sqrt{b} + C\sqrt{c} = - \left(4\sqrt{ab} + \sqrt{ac} + 4\sqrt{bc} + D\right).$$

Squaring both sides, we get

$$(2.5) \quad E\sqrt{ab} + F\sqrt{ac} = G\sqrt{bc} + H,$$

where

$$E := 2AB - 8D - 8c, \\ F := 2AC - 2D - 32b, \\ G := 8D + 8a - 2BC, \\ H := D^2 + 16ab + ac + 16bc - aA^2 - bB^2 - cC^2.$$

Squaring again, we conclude from (2.5) that

$$4(aEF - GH)^2 bc = (bcG^2 + H^2 - abE^2 - acF^2)^2.$$

The final computation is left to a computer. Putting in all the expressions for  $A, B, \dots, H$  and for  $a, b, c$ , we get the polynomial  $P$  described in Theorem 1.1.

## NOTE ADDED IN PROOF

In the context of Theorem 1.1 one may replace Definition 1.1 by the following more convenient definition: A real function  $y = y(x)$  with domain  $\mathbb{R}$  is said to be a *local solution of a functional equation of degree 3*, if there is a polynomial  $P(u_1, \dots, u_9)$  such that for every point  $x_1$  from  $\mathbb{R}$  and for some sufficiently small  $\varepsilon > 0$  depending on  $x_1$  the identity

$$P\left(y'(x_1), y'\left(\frac{x_1 + x_2}{2}\right), y'(x_2), y''(x_1), y''\left(\frac{x_1 + x_2}{2}\right), \dots, y'''(x_2)\right) = 0$$

holds for every real  $x_2$  with  $0 \leq x_2 - x_1 \leq \varepsilon$ .

Instead of (1.5) put:

$$y_1 := y(x_1), \quad y_2 := y\left(\frac{x_1 + x_2}{2}\right), \quad y_3 := y(x_2).$$

Then, Theorem 1.1 remains true, and there are no essential changes within the proof.

## REFERENCES

1. M. Boshernitzan, *Universal formulae and universal differential equations*, Annals of Mathematics **124** (1986), 273–291. MR **88a**:12007
2. C. Elsner, *On the approximation of continuous functions by  $C^\infty$ -solutions of third-order algebraic differential equations*, Math. Nachr. **157** (1992), 235–241. MR **94i**:34032
3. L. A. Rubel, *A universal differential equation*, Bulletin of the American Mathematical Society **4** no. 3 (1981), 345–349. MR **82e**:34015

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HANNOVER, WELFENGARTEN 1, D-30167 HANNOVER, GERMANY

*E-mail address*: `elsner@math.uni-hannover.de`