PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 2, February 1999, Pages 569–575
S 0002-9939(99)04484-6

L^p WILLMORE FUNCTIONALS

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(Communicated by Christopher Croke)

ABSTRACT. We prove some integral inequalities for immersed tori in the three sphere. The functionals considered are generalizations of the Willmore functional.

1. INTRODUCTION

Let M be an immersed torus in S^3, the standard unit sphere. Denote by κ_1 and κ_2 the principal curvatures of M and assume that $\kappa_1 \geq \kappa_2$. Define

$$W_p = \int_M (\frac{\kappa_1 - \kappa_2}{2})^p \, dM$$

where dM is the area element of M.

In this paper, we prove the following

Theorem 1.1. *For $p \geq 1 + \frac{\pi}{2}$, $W_p \geq 2\pi^2$, and equality holds if and only if M is the Clifford torus.*

The Willmore conjecture says that $W_2 \geq 2\pi^2$. The conjecture has been proved for some conformal classes by Li and Yau [LY], Montiel and Ros [MR], and R. Bryant (see the note added in proof in [MR]). The conjecture is also known to be true for surfaces whose images under stereographic projection are surfaces of revolution in $\mathbf{R}^3$ [LS]. For W_2, we will prove the following

Theorem 1.2. $W_2 + \int_M ((1 - K_-)^{\frac{1}{2}+\frac{\pi}{4}} - (1 - K_-)) \geq 2\pi^2$*, and equality holds if and only if M is the Clifford torus, where $K_- = \min\{K, 0\}$ and K is the Gauss curvature of M.*

In particular, Theorem 1.2 implies that the Willmore conjecture holds for flat tori, a theorem due to B.Y. Chen [C] (also see [LY] for a different proof). In [P], Pinkall showed that every torus can be conformally imbedded into S^3 as a flat torus (a Hopf torus). In a separate paper, we will show that the Willmore conjecture also holds for tori near flat tori.

We note that L. Simon [S] has shown that there exists an embedded torus minimizing W_2. For $p > 2$, similar functionals have been considered by Langer [Lan]

Received by the editors January 30, 1997 and, in revised form, May 19, 1997.
1991 *Mathematics Subject Classification.* Primary 53C20.
Key words and phrases. Principal curvature, exponential map.

and Hutchinson [H]. They considered the L^p norm of the second fundermental form, namely,

$$E_p = \int_M (\kappa_1^2 + \kappa_2^2)^{\frac{p}{2}}\, dM$$

and proved compactness theorems for E_p.

2. Proof of theorems

Theorems 1.1 and 1.2 are consequences of the following.

Theorem 2.1.

$$\int_M \frac{\kappa_1 - \kappa_2}{2} + \frac{1}{2}(1 + \kappa_1\kappa_2)(\arctan\kappa_2 - \arctan\kappa_1)\, dM \ \geq 2\pi^2.$$

Remark 2.2. By the Gauss equation, the Gauss curvature of M, $K = 1 + \kappa_1\kappa_2$. The Willmore conjecture for flat tori is then an immediate consequence of Theorem 2.1 since, in this case,

$$\frac{\kappa_1 - \kappa_2}{2} = \frac{1}{2}(\kappa_1 + \frac{1}{\kappa_1}) \leq [\frac{1}{2}(\kappa_1 + \frac{1}{\kappa_1})]^2 = (\frac{\kappa_1 - \kappa_2}{2})^2.$$

Proof. For $t \geq 0$, denote by M_+^t the equidistant surface of M in the normal direction N, i.e.

$$M_+^t = \{exp_x tN | x \in M\}.$$

Let $A_+(x,t)$ be the area element of M_+^t. $A_+(x,t)$ is well-defined and differentiable for t less than the first focal distance in the direction of N and the first variation formula gives

$$\frac{\partial A_+}{\partial t} = H(x,t)A_+(x,t) \tag{2.1}$$

where $H(x,t)$ is the mean curvature of M with respect to N, i.e. the sum of two principal curvatures.

Let $\lambda_1(x,t)$ and $\lambda_2(x,t)$ be the principal curvatures of M_+^t, where $\lambda_i(x,0) = \kappa_i$, $i = 1, 2$.

By the Ricatti equation for the second fundamental form (see e.g. [CG] for a derivation),

$$\frac{\partial}{\partial t}\lambda_i = -1 - \lambda_i^2.$$

Solving it, we get

$$\lambda_i(x,t) = \cot(t + \alpha_i)$$

where $\alpha_i = \operatorname{arccot}(\kappa_i), i = 1, 2$.

Substituting $\cot(t + \alpha_1) + \cot(t + \alpha_2)$ for H in (2.1) and solving the resulting equation yields

$$A_+(x,t) = \csc(\alpha_1)\csc(\alpha_2)\sin(t + \alpha_1)\sin(t + \alpha_2).$$

Similarly, we consider the equidistant surfaces of M in the direction of $-N$, M^t_-, and get

$$A_-(x,t) = \csc(\alpha_1)\csc(\alpha_2)\sin(t-\alpha_1)\sin(t-\alpha_2)$$

where $A_-(x,t)$ is the area element of M^t_-.

Since $\alpha_1 \le \alpha_2$, it is clear that the first zero of $A_+(x,t)$ ($A_-(x,t)$, respectively) occurs at $t = \pi - \alpha_2$ (α_1, respectively). Hence the first focal distance in the direction of N ($-N$, respectively) is less than or equal to $t = \pi - \alpha_2$ (α_1, respectively). For each point p in S^3, there exists a geodesic from p to M which hits M perpendicularly and minimizes the distance from p to M. The length of the geodesic is less than or equal to the first focal distance in the direction of N or $-N$. This implies that

$$S^3 = S^3_+ \bigcup S^3_-$$

where,

$$\begin{aligned} S^3_+ &= \{exp_x(tN) | x \in M, 0 \le t \le \pi - \alpha_2\} \quad \text{and} \\ S^3_- &= \{exp_x(-tN) | x \in M, 0 \le t \le \alpha_1\}. \end{aligned}$$

Now,

$$\begin{aligned} \mathrm{vol}(S^3_+) &\le \int_M \int_0^{\pi-\alpha_2} A_+(x,t)\,dt\,dx \\ &= \frac{1}{2}\int_M \int_0^{\pi-\alpha_2} \csc\alpha_1 \csc\alpha_2[\cos(\alpha_2-\alpha_1) - \cos(2t+\alpha_1+\alpha_2)]\,dt\,dx \\ &= \frac{1}{2}\int_M [(\pi-\alpha_2)(1+\cot\alpha_1\cot\alpha_2) + \cot\alpha_1]\,dx \end{aligned}$$

and

$$\mathrm{vol}(S^3_-) \le \int_M \int_0^{\alpha_1} A_-(x,t)\,dt\,dx = \frac{1}{2}\int_M [\alpha_1(1+\cot\alpha_1\cot\alpha_2) - \cot\alpha_2]\,dx.$$

It then follows that

$$\begin{aligned} 2\pi^2 &= \mathrm{vol}(S^3) \le \mathrm{vol}(S^3_+) + \mathrm{vol}(S^3_-) \\ &\le \frac{1}{2}\int_M [(\pi-\alpha_2)(1+\cot\alpha_1\cot\alpha_2) + \cot\alpha_1]\,dx \\ &\quad + \frac{1}{2}\int_M [\alpha_1(1+\cot\alpha_1\cot\alpha_2) - \cot\alpha_2]\,dx \\ &= \int_M [\frac{\cot\alpha_1 - \cot\alpha_2}{2} + \frac{1}{2}(1+\cot\alpha_1\cot\alpha_2)(\pi+\alpha_1-\alpha_2)]\,dM \\ &= \int_M [\frac{\kappa_1-\kappa_2}{2} + \frac{1}{2}(1+\kappa_1\kappa_2)(\arctan\kappa_2 - \arctan\kappa_1)]\,dM \end{aligned}$$

where we have used the Gauss-Bonnet formula. This proves Theorem 2.1. $\square$

Let $d \ge 0$, $c \ge 0$ be fixed constants and define

$$W(r) = d + (1 + r(r+d))[\arctan r - \arctan(r+d) - 2c].$$

Lemma 2.3. *$W(r) \le d + 2(\frac{d^2}{4} - 1)(\arctan\frac{d}{2} + c)$, for $-\infty < r < \infty$, and equality holds if and only if $r = -\frac{d}{2}$.*

Proof. Taking derivative of W gives

$$W'(r) = (2r + d)Z(r)$$

where, $Z(r) = \dfrac{d(1 + r(r + d))}{(1 + r^2)(1 + (r + d)^2)} + \arctan(r) - \arctan(r + d) - 2c$. Since

$$Z'(r) = 2d^3 \frac{2r + d}{(1 + r^2)^2(1 + (r + d)^2)^2},$$

we see that $Z(r) \downarrow$ for $r \le -\frac{d}{2}$ and $Z(r) \uparrow$ for $r \ge -\frac{d}{2}$. Notice that $\lim_{r\to\infty} Z(r) = \lim_{r\to-\infty} Z(r) = -2c \le 0$. Hence, $Z(r) < 0$ for all r. This implies that $W(r)$ is strictly increasing for $r \le -\frac{d}{2}$ and strictly decreasing for $r \ge -\frac{d}{2}$. It then follows that $r = -\frac{d}{2}$ is the only maximum of $W(r)$, and consequently, $W(r) \le W(-\dfrac{d}{2}) = d + 2(\dfrac{d^2}{4} - 1)(\arctan \dfrac{d}{2} + c)$, for $-\infty < r < \infty$ and equality holds when and only when $r = -\dfrac{d}{2}$. Lemma 2.3 is thus proved. □

Lemma 2.3, combined with Theorem 2.1, gives

Lemma 2.4. $\int_M [\dfrac{d}{2} + (\dfrac{d^2}{4} - 1)(\arctan \dfrac{d}{2} + c)]\, dM \ge 2\pi^2$, *where* $d = \kappa_1 - \kappa_2$ *and* $c \ge 0$ *is an arbitrary constant. Moreover, when the equality holds,* M *is minimal.*

Proof. We have from Theorem 2.1 and the Gauss-Bonnet formula that for any constant $c \ge 0$,

$$\int_M \frac{\kappa_1 - \kappa_2}{2} + \frac{1}{2}(1 + \kappa_1\kappa_2)(\arctan \kappa_2 - \arctan \kappa_1 - 2c)\, dM \;\ge 2\pi^2. \tag{2.2}$$

Choosing $d = \kappa_1 - \kappa_2$ and $r = \kappa_2$ in Lemma 2.3, we get

$$\begin{aligned} \int_M [\frac{\kappa_1 - \kappa_2}{2} \quad &+ \quad \frac{1}{2}(1 + \kappa_1\kappa_2)(\arctan \kappa_2 - \arctan \kappa_1 - 2c)]\, dM \\ &\le \quad \int_M [\frac{d}{2} + (\frac{d^2}{4} - 1)(\arctan \frac{d}{2} + c)]\, dM. \end{aligned} \tag{2.3}$$

The inequality in Lemma 2.4 now follows from (2.2) and (2.3). If the equality holds, then the equality in (2.3) also holds. By Lemma 2.3, the equality in (2.3) implies that $\kappa_2 = -\frac{\kappa_1 - \kappa_2}{2}$, i.e. $H = 0$. Lemma 2.4 is thus proved. □

Theorem 1.1 will be a consequence of Lemma 2.4 and the following elementary lemma.

Lemma 2.5. *For any* $p \ge 1 + \frac{\pi}{2}$,

$$Q_p(t) = t + (t^2 - 1)(\arctan t + \frac{1}{2}(p - 1 - \frac{\pi}{2})) - t^p \le 0, \qquad t \ge 0,$$

and the only possible points where the equality holds are $t = 0$ *and* $t = 1$.

Proof. We first show that $Q_p(t) < 0$ for $t > 1$. Since $Q_p(1) = 0$, it suffices to show that $Q_p'(t) < 0$ for $t > 1$. Now,

$$Q_p'(t) = 2tR_p(t),$$

where

$$R_p(t) = \frac{t}{1+t^2} - \frac{p}{2}t^{p-2} + \arctan t + \frac{1}{2}(p - 1 - \frac{\pi}{2}).$$

Taking the derivative of R_p gives

$$R_p'(t) = 2t^{p-3}S(t)$$

where

$$S(t) = \frac{t^{3-p}}{(1+t^2)^2} - \frac{p(p-2)}{4}.$$

The derivative of $S(t)$ is

$$S'(t) = \frac{(p+1)t^{2-p}}{(1+t^2)^3}(\frac{3-p}{1+p} - t^2). \tag{2.4}$$

Since $\dfrac{3-p}{1+p} < 1$, we find that $S'(t) < 0$ for $t > 1$. So $S(t) \leq S(1) = \dfrac{1}{4}(1-p(p-2)) < 0$, for $t > 1$, and hence $R_p(t) < R(1) = 0$ for $t > 1$, therefore $Q_p(t) < Q(1) = 0$ for $t > 1$.

To prove $Q_p(t) < 0$ for $0 < t < 1$, we first show that for each fixed $t : 0 < t \leq 1$, $Q_p(t)$ is a decreasing function in p. In fact,

$$\frac{\partial Q}{\partial p} = \frac{1}{2}(t^2 - 1) - t^p \ln t = \frac{1}{2}t^p(t^{2-p} - t^{-p} - 2\ln t).$$

Let $a(t) = t^{2-p} - t^{-p} - 2\ln t$. It is easy to see that $a(t)$ is an increasing function in t on $(0, 1)$ and $a(1) = 0$. So $a(t) \leq 0$ and hence $Q_p(t)$ is a decreasing function in p. Now all we need to prove is that $Q_{1+\frac{\pi}{2}}(t) < 0$ for $0 < t < 1$.

It is easy to see that

(1°) $Q_{1+\frac{\pi}{2}}(0) = 0$ and $Q_{1+\frac{\pi}{2}}(t)$ is decreasing near 0 .

(2°) $Q_{1+\frac{\pi}{2}}(1) = 0$, $Q'_{1+\frac{\pi}{2}}(1) = 0$ and $Q''_{1+\frac{\pi}{2}}(1) < 0$, i.e. 1 is a local maximum of $Q_{1+\frac{\pi}{2}}$.

We can also prove that

(3°) $Q_{1+\frac{\pi}{2}}$ has only one critical point on $(0, 1)$, i.e. $R_{1+\frac{\pi}{2}}$ has only one zero on $(0, 1)$.

In fact, it is clear from (2.4) that $S'(t)$ has at most one zero on $(0, \infty)$, so $S(t)$ has at most two zeros by Rolle's Lemma, hence (a) $R_{1+\frac{\pi}{2}}(t)$ has at most two critical points on $(0, \infty)$. We can also easily see that (b) $R_{1+\frac{\pi}{2}}(0) = 0$, $R'_{1+\frac{\pi}{2}}(t) \to -\infty$, as $t \to 0^+$, and that (c) $R_{1+\frac{\pi}{2}}(1) = 0$, $R'_{1+\frac{\pi}{2}}(1) < 0$. We then conclude from (a), (b) and (c) that $R_{1+\frac{\pi}{2}}$ has only one zero on $(0, 1)$. This proves (3°). (1°), (2°) and (3°) imply that $Q_{1+\frac{\pi}{2}}(t) < 0$ for $0 < t < 1$. The proof of Lemma 2.5 is then completed. □

Proof of Theorem 1.1. Lemma 2.5 implies that

$$\int_M [\frac{d}{2} + (\frac{d^2}{4} - 1)(\arctan\frac{d}{2} + \frac{1}{2}(p - 1 - \frac{\pi}{2}))]\, dM \leq \int_M (\frac{\kappa_1 - \kappa_2}{2})^p\; dM \tag{2.5}$$

where $d = \kappa_1 - \kappa_2$.

The inequality in Theorem 1.1 now follows from (2.5) and Lemma 2.4 in which we choose $c = \frac{1}{2}(p - 1 - \frac{\pi}{2})$.

If the equality in Theorem 1.1 holds, then equalities in both (2.5) and Lemma 2.4 hold. By Lemma 2.5, the equality in (2.5) implies that

$$\frac{\kappa_1-\kappa_2}{2}\equiv 0 \text{ or } \frac{\kappa_1-\kappa_2}{2}\equiv 1.$$

Since M is a torus, we cannot have $\frac{\kappa_1-\kappa_2}{2}\equiv 0$ by the Gauss-Bonnet formula, hence we must have

$$\frac{\kappa_1-\kappa_2}{2}\equiv 1. \tag{2.6}$$

On the other hand, that the equality in Lemma 2.4 holds implies that M is minimal, i.e.

$$\kappa_1+\kappa_2=0. \tag{2.7}$$

(2.6) and (2.7) imply that $\kappa_1=1$ and $\kappa_2=-1$, hence M is both minimal and flat. By a theorem due to Lawson [Law], the only minimal flat torus in S^3 is the Clifford torus. This proves Theorem 1.1.

We are now in position to prove Theorem 1.2.

We first state a lemma whose proof is similar to that of Lemma 2.3 and hence will be omitted here.

Lemma 2.6. *Let D be a constant and $f(t)$ be defined as*

$$f(t)=\frac{1}{2}(t-\frac{D}{t})+\frac{1}{2}(1+D)(\arctan\frac{D}{t}-\arctan t)-\frac{1}{4}(t-\frac{D}{t})^2.$$

1. *If $D>0$, then $f(t)\le 0$ for $t\neq 0$.*
2. *If $D<0$, then $f(t)\le f(\sqrt{-D})$ for $t\neq 0$, and $f(t)=f(\sqrt{-D})$ only when $t=\sqrt{-D}$.*
3. *If $D=0$, then $f(t)\le 0$ for all t.*

Choosing $D=K-1(=\kappa_1\kappa_2)$ and $t=\kappa_1$ in Lemma 2.6, we get from Theorem 2.1 the following

Lemma 2.7.

$$W_2+\int_{\{x\in M|K(x)<1\}}[\sqrt{1-K}-K\arctan\sqrt{1-K}-(1-K)]\,dM\ge 2\pi^2,$$

and when equality holds, M is minimal (and hence $K<1$).

Notice that $\sqrt{1-K}-K\arctan\sqrt{1-K}-(1-K)\le 0$ when $0\le K\le 1$, as it can be shown by an argument similar to the proof of Lemma 2.5. It then follows from Lemma 2.5 (with $p=1+\frac{\pi}{2}$) that

$$\begin{aligned}\int_{\{x\in M|K(x)<1\}}[\sqrt{1-K} \quad &- \quad K\arctan\sqrt{1-K}-(1-K)]\,dM\\ &\le \int_{\{x\in M|K(x)\le 0\}}[(1-K)^{\frac{1}{2}+\frac{\pi}{4}}-(1-K)]\,dM\\ &= \int_M[(1-K_-)^{\frac{1}{2}+\frac{\pi}{4}}-(1-K_-)]\,dM. \end{aligned}\tag{2.8}$$

The above equality holds if and only if $K\le 0$. This in turn implies that the equality holds if and only if $K\equiv 0$, by Lemma 2.5.

The inequality in Theorem 1.2 now follows from Lemma 2.7 and (2.8). If the equality in Theorem 1.2 holds, then equalities in both Lemma 2.7 and (2.8) hold. The equality in Lemma 2.7 implies that M is minimal, while the equality in (2.8) implies that $K \equiv 0$. Hence M is minimal and flat; again by Lawson's theorem, M must be the Clifford torus. Theorem 1.2 is thus proved.

I would like to thank Prof. Greg Galloway for many inspiring discussions. I would also like to thank Prof. Bill Minicozzi for sharing his insights and for informing me papers [Lan] and [H].

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