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ON A THEOREM OF E. HELLY

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ABSTRACT. E. Helly's theorem asserts that *any bounded sequence of monotone real functions contains a pointwise convergent subsequence.* We reprove this theorem in a generalized version in terms of monotone functions on linearly ordered sets. We show that the cardinal number responsible for this generalization is exactly the splitting number. We also show that a positive answer to a problem of S. Saks is obtained under the assumption of the splitting number being strictly greater than the first uncountable cardinal.

0. INTRODUCTION

E. Helly's theorem ([3]) asserts that *any bounded sequence of monotone real functions contains a pointwise convergent subsequence.* In the present paper, we prove the following generalization of the theorem: *for linearly ordered sets X and Y, if Y is sequentially compact with density less than the splitting number* $\mathbf{s}$, *then any sequence of monotone functions from X to Y contains a pointwise convergent subsequence* (Theorem 7). We also show that this theorem characterizes the splitting number (Theorem 9).

We begin with reviewing some definitions and elementary facts needed for our results.

1. PRELIMINARIES: LINEARLY ORDERED SETS

A linearly ordered set X is said to be *dense linear order*, if, for any $x, y \in X$, $x < y$ implies that there exists $z \in X$ such that $x < z < y$. A subset D of a linearly ordered set X is said to be *dense in* X, if, for any $x, y \in X$, $x < y$ implies that there exists $z \in D$ such that $x \leq z \leq y$. The *density* of X is defined by

$$d(X) = \min\{\, |D| \,:\, D \subseteq X \text{ and } D \text{ is dense in } X\,\},$$

where $|D|$ denotes the cardinality of the set D.

Let X and Y be linearly ordered sets. A function $f : X \to Y$ is said to be *increasing,* if, for any $x, y \in X$, $x < y$ implies $f(x) \leq f(y)$; *decreasing,* if $x < y$ implies $f(y) \leq f(x)$. A function is monotone if it is either increasing or decreasing. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is called increasing, decreasing or monotone respectively, if it is increasing, decreasing or monotone respectively as a function from the set of all natural numbers $\mathbb{N}$ into X.

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For a linearly ordered set X, the notion of convergence can be introduced in a canonical way: an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to a point x, if x is the supremum of the set of all elements of this sequence; a decreasing sequence converges to x, if x is the infimum of the set of all elements of this sequence; in general, a sequence $(x_n)_{n\in\mathbb{N}}$ *converges to* x, if every monotone subsequence of this sequence converges to x. We say also that a sequence is *convergent* if it converges to some x. A linearly ordered set is said to be *sequentially compact* if each monotone sequence of its elements converges to some point in it. If a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x, we denote this as usual by $\lim_{n\to\infty} x_n = x$. For a sequence $(x_n)_{n\in I}$ indexed by an infinite subset I of $\mathbb{N}$, its convergence to a point x is defined similarly and denoted by $\lim_{n\in I} x_n = x$. For an infinite $I \subseteq \mathbb{N}$ and a sequence $(f_n)_{n\in I}$ of functions from a set X to a linearly ordered set Y, we say that $(f_n)_{n\in I}$ *converges pointwise* to $f : X \to Y$, if $\lim_{n\in I} f_n(x) = f(x)$ holds for every $x \in X$. We shall also say that a sequence $(f_n)_{n\in I}$ is pointwise convergent if there is some function f to which the sequence converges pointwise.

Since any sequence in a linearly ordered set has a monotone subsequence, every sequence in a sequentially compact linearly ordered set has a convergent subsequence. Using this fact, we can see easily the following.

Lemma 1. *If $(x_n)_{n\in\mathbb{N}}$ is a non-convergent sequence of elements of a sequentially compact linearly ordered set X, then there exist infinite subsets of natural numbers I and J such that subsequences $(x_n)_{n\in I}$ and $(x_n)_{n\in J}$ converge to different points of X.* □

Lemma 2. *Any infinite linearly ordered set X can be embedded into a dense linear order $\tilde{X}$ such that $d(\tilde{X}) = d(X)$. If X is sequentially compact, then $\tilde{X}$ can be also chosen to be so. Also, convergent sequences in X remain convergent in $\tilde{X}$ with the same limit.*

Proof. Let D be a dense subset of X of cardinality $d(X)$. For points $x, y \in X$, let us call (x, y) a jump in X if $x < y$ and there is no $z \in X$ such that $x < z < y$. By definition of dense subsets, for each jump (x, y) one of the points x or y must be in D. Hence there are at most $d(X)$ jumps. Let $\tilde{X}$ be the linearly ordered set constructed from X by inserting a copy of the reals into each of the jumps in X. Noting that the density of the reals $\mathbb{R}$, with respect to the canonical ordering, is countable and $\mathbb{R} \cup \{-\infty, +\infty\}$ is sequentially compact, it is easy to see that $\tilde{X}$ is as desired. □

Lemma 3. *Suppose that X and Y are linearly ordered sets and $(f_n)_{n\in I}$ is a sequence of increasing functions from X to Y. If $(f_n)_{n\in I}$ converges pointwise to a function $f : X \to Y$, then f is also increasing.*

Proof. By Lemma 2, we may assume that Y is a dense linear order. The rest of the proof can be done just like the usual proof of the corresponding assertion on increasing real functions. □

Any linearly ordered set X can be densely embedded into a sequentially compact linearly ordered set $\overline{X}$. E.g. we can take the Dedekind completion of X as $\overline{X}$. Note that we have $d(X) = d(\overline{X})$ since X is dense in $\overline{X}$. In general, the Dedekind completion of X is not the minimal sequentially compact linearly ordered set containing a dense copy of X since there can be an unfilled Dedekind cut (D, E) of X such that D has uncountable cofinality and E uncountable coinitiality. Let us call a sequence

$(x_n)_{n\in\mathbb{N}}$ in a linearly ordered set X *potentially convergent* if it converges to some point in some $\overline{X}$ as above. By virtue of Lemma 1, this is equivalent to saying that there are no $x,\ y \in X$ and no infinite $I,\ J \subseteq \mathbb{N}$ such that $x_n \le x < y \le x_m$ for every $n \in I$ and $m \in J$.

2. The splitting number

A family $\mathcal{S}$ of infinite subsets of $\mathbb{N}$ is said to be *splitting* if, for every infinite subset $I \subseteq \mathbb{N}$, there exists a set $J \in \mathcal{S}$ such that $I \cap J$ and $I \setminus J$ are both infinite. The splitting number $\mathbf{s}$ is defined by

$$\mathbf{s} = \min\{\, |\mathcal{S}| \,:\, \mathcal{S} \text{ is a splitting family}\,\}.$$

In particular, if $\mathcal{S}$ is a family of infinite subsets of $\mathbb{N}$ of cardinality less than $\mathbf{s}$, then there exists an infinite subset $I \subseteq \mathbb{N}$ such that I is almost included either in J or in $\mathbb{N} \setminus J$ for every $J \in \mathcal{S}$. It is readily seen that $\mathbf{s}$ is uncountable and less than or equal to the cardinality of the reals. On the other hand, it is known that the value of $\mathbf{s}$ cannot be decided from the axioms of set theory alone. A splitting family was considered first by Sierpiński in [5]. He showed that under the Continuum Hypothesis there is a splitting family $\mathcal{S}$ with the property that every uncountable subfamily of $\mathcal{S}$ is still splitting. For more about the cardinal $\mathbf{s}$ and its relation to other cardinal invariants of reals the reader may consult [1], [2] or [7]. The role of splitting number in connection with convergence was also studied in [8].

The following lemma is the set-theoretic core of the generalization of Helly's theorem.

Lemma 4. *If X is a set of cardinality less than $\mathbf{s}$ and Y is a sequentially compact linearly ordered set of density less than $\mathbf{s}$, then for any sequence $(f_n)_{n\in\mathbb{N}}$ of functions from X to Y there exists an infinite subset $I \subseteq \mathbb{N}$ such that the sequence of functions $(f_n)_{n\in I}$ converges pointwise.*

Proof. By Lemma 2 we may assume that Y is a dense linear order. Let D be a dense subset of Y of cardinality less than $\mathbf{s}$. For $x \in X$ and $y \in D$, let

$$C_x^y = \{\, n \in \mathbb{N} \,:\, f_n(x) < y \,\}.$$

Since $|X \times D| < \mathbf{s}$, there exists an infinite $I \subseteq \mathbb{N}$ such that I is almost included either in C_x^y or in $\mathbb{N} \setminus C_x^y$ for any $x \in X$ and $y \in D$.

We shall show that the set I is as desired. Otherwise there would be some point $a \in X$ such that the sequence $(f_n(a))_{n\in I}$ of points in Y is not convergent. Then, by Lemma 1, there are infinite subsets J and K of I and a point $d \in D$ such that sequences $(f_n(a))_{n\in J}$ and $(f_n(a))_{n\in K}$ of points in Y are convergent and we have

$$\lim_{n\in J} f_n(a) < d < \lim_{n\in K} f_n(a).$$

Hence we have $f_n(a) < d$ for all but finitely many $n \in J$ and $d < f_n(a)$ for all but finitely many $n \in K$. It follows that the sets $I \cap C_a^d$ and $I \setminus C_a^d$ are both infinite; but this contradicts the choice of I. $\square$

Lemma 4 gives the consistency of a positive answer to the following question of S. Saks studied in [5]:

For arbitrary sequence $(f_n)_{n\in\mathbb{N}}$ of real functions, do there exist an infinite $I \subseteq \mathbb{N}$ and an uncountable $X \subset \mathbb{R}$ such that, for each $x \in X$, the sequence of real numbers $(f_n(x))_{n\in I}$ has a finite or infinite limit?

Under the Continuum Hypothesis, Sierpiński gave a negative answer to the question in [5]. By applying Lemma 4 for the sequentially compact linearly ordered set $\mathbb{R} \cup \{-\infty, +\infty\}$, we see that, under $\mathbf{s} > \aleph_1$, a positive answer to the question is obtained.

Since every linearly ordered set can be embedded densely into a sequentially compact linearly ordered set, the next lemma follows immediately from Lemma 4.

Lemma 5. *If X is a set of cardinality less than $\mathbf{s}$ and Y is a linearly ordered set of density less than $\mathbf{s}$, then for any sequence $(f_n)_{n\in\mathbb{N}}$ of functions from X to Y there exists an infinite subset $I \subseteq \mathbb{N}$ such that the sequence $(f_n(x))_{n\in I}$ is potentially convergent for every $x \in X$.* □

Lemma 5 can be yet slightly improved. For any infinite $I \subseteq \mathbb{N}$, let us call a sequence $(x_n)_{n\in I}$ in a linearly ordered set X *semi-monotone* if there is a bijection $\varphi : \mathbb{N} \to I$ such that $(x_{\varphi(n)})_{n\in\mathbb{N}}$ is eventually monotone, i.e. monotone from some $m \in \mathbb{N}$ on. It is clear that a semi-monotone sequence is potentially convergent. If $x = \lim_{n\in I} x_n$ exists, then $(x_n)_{n\in I}$ is semi-monotone if and only if $(x_n)_{n\in I}$ approaches to x eventually from one side — i.e. for some $m \in \mathbb{N}$ either $x_n \leq x$ for every $n \geq m$ or $x \leq x_n$ for every $n \geq m$.

Lemma 6. *If X is a set of cardinality less than $\mathbf{s}$ and Y is a linearly ordered set of density less than $\mathbf{s}$, then for any sequence $(f_n)_{n\in\mathbb{N}}$ of functions from X to Y there exists an infinite subset $I \subseteq \mathbb{N}$ such that the sequence $(f_n(x))_{n\in I}$ is semi-monotone for every $x \in X$.*

Proof. Without loss of generality, we may assume that Y is sequentially compact. By Lemma 4, there is an infinite $I \subseteq \mathbb{N}$ such that $(f_n)_{n\in I}$ is pointwise convergent. For each $x \in X$ let $y_x = \lim_{n\in I} f_n(x)$. Let $\tilde{Y}$ be the linearly ordered set obtained from Y by inserting a new point y'_x between y_x and $\{\, y \in Y \,:\, y_x < y \,\}$ for each $x \in X$. $\tilde{Y}$ is still sequentially compact and $d(\tilde{Y}) < \mathbf{s}$ since only fewer than $\mathbf{s}$ new points are added. Hence we can apply Lemma 4 again to $(f_n)_{n\in I}$ as a sequence of functions from X to $\tilde{Y}$ to obtain an infinite $J \subseteq I$ such that $(f_n)_{n\in J}$ is pointwise convergent as a sequence of functions from X to $\tilde{Y}$. For each $x \in X$, as $(f_n(x))_{n\in J}$ should converge to y_x or y'_x, it follows that, for each $x \in X$, $(f_n(x))_{n\in J}$ as a sequence of points in Y approaches y_x eventually from one side. Hence by the remark before this lemma, $(f_n(x))_{n\in J}$ is a quasi-monotone sequence in Y. □

3. Generalized Helly's theorem

Since the density of the reals is countable, Helly's theorem ([3]) as cited in the Introduction is just a special case of the following theorem.

Theorem 7 (Generalized Helly's Theorem)**.** *Let X and Y be linearly ordered sets. If Y is sequentially compact with density less than $\mathbf{s}$, then any sequence of monotone functions from X to Y contains a pointwise convergent subsequence.*

Proof. Without loss of generality, we may assume that the sequence $(f_n)_{n\in\mathbb{N}}$ consists of increasing functions. By Lemma 2 we may also assume that Y is a dense linear order. Let $D \subseteq Y$ be a dense subset of cardinality less than $\mathbf{s}$. For b, $d \in D$ and $n \in \mathbb{N}$, let $x^n_{b,d}$ be an element of X such that $b \leq f_n(x^n_{b,d}) \leq d$ if such an element exists; otherwise let $x^n_{b,d}$ be an arbitrary element of X. Let

$$Z = \{\, x^n_{b,d} \,:\, b, d \in D,\, n \in \mathbb{N} \,\}.$$

Then we have $|Z| < \mathbf{s}$. Hence, by Lemma 4, there exists an infinite subset $L \subseteq \mathbb{N}$ such that the sequence of functions $(f_n \restriction Z)_{n\in L}$ converges pointwise where $f_n \restriction Z$ denotes the restriction of the function f_n to the set Z. Let

$$T = \{\, x \in X \,:\, (f_n(x))_{n\in L} \text{ is a convergent sequence of points in } Y \,\}.$$

We have $Z \subseteq T$. Let $h : T \to Y$ be the function such that the sequence of functions $(f_n \restriction T)_{n\in L}$ converges pointwise to h. By Lemma 3, h is an increasing function. Now let

$$\mathcal{U} = \{\, U \,:\, U \text{ is a maximal interval in } X \text{ such that } U \subseteq X \setminus T \,\}.$$

For each interval $U \in \mathcal{U}$ we choose $x_U \in U$. By definition of Z, f_n is constant on each $U \in \mathcal{U}$ for every $n \in \mathbb{N}$. Hence, for any subset $M \subseteq L$, we have:

$$(*) \qquad (f_n(x_U))_{n\in M} \text{ converges if and only if } (f_n \restriction U)_{n\in M} \text{ converges pointwise.}$$

Letting $W = \{\, x_U \,:\, U \in \mathcal{U} \,\}$, we claim that $|W| < \mathbf{s}$. To see this, let $x \in W$. The sequence $(f_n(x))_{n\in L}$ is not convergent. Since Y is a dense linear order and sequentially compact, by Lemma 1, there are infinite subsets J, $K \subseteq L$ and points b_x, c_x, $d_x \in D$ such that sequences of points $(f_n(x))_{n\in J}$ and $(f_n(x))_{n\in K}$ are convergent and

$$\lim_{n\in J} f_n(x) < b_x < d_x < c_x < \lim_{n\in K} f_n(x).$$

If $y \in T$ and $y < x$, then $h(y) \leq b_x$, since $\lim_{n\in L} f_n(y) = h(y)$ and $f_n(y) < b_x$ for infinitely many $n \in L$. Likewise, for any $z \in T$ with $x < z$ we have $h(z) \geq c_x$. For $x_1, x_2 \in W$ with $x_1 < x_2$, there is $y \in T$ such that $x_1 < y < x_2$. Hence the mapping from W to D defined by $x \mapsto d_x$ is injective. As $|D| < \mathbf{s}$, it follows that $|W| < \mathbf{s}$.

Again by Lemma 4 we can find an infinite $I \subseteq L$ such that $(f_n \restriction W)_{n\in I}$ is pointwise convergent. By definitions and $(*)$ above, we have that the sequence of functions $(f_n)_{n\in I}$ converges pointwise. □

4. The splitting number is optimal

For an infinite subset $V \subseteq \mathbb{N}$, let

$$\mathcal{D}(V) = \sum_{n\in V} \frac{1}{2^{n+1}}.$$

Note that $\mathcal{D}$ is a bijective mapping from infinite subsets of $\mathbb{N}$ to the real numbers in the half-open interval $(0, 1]$. For a family $\mathcal{S}$ of subsets of $\mathbb{N}$, let us denote by $\mathcal{D}(\mathcal{S})$ the set $\{\, \mathcal{D}(V) \,:\, V \in \mathcal{S} \,\}$. Thus $\mathcal{D}(\mathcal{S})$ is a subset of the unit interval of cardinality $|\mathcal{S}|$.

Assume now that $\mathcal{S}$ is a splitting family of cardinality $\mathbf{s}$. Let

$$H = (\mathcal{D}(\mathcal{S}) \times [0, 1]) \cup ([0, 1] \times \{0\})$$

be the linearly ordered set equipped with the lexicographical ordering, i.e. we let $(x, y) < (p, q)$, whenever $x < p$, or $x = p$ and $y < q$. Here, $x < p$ and $y < q$ denote the canonical ordering on the reals.

Lemma 8. *The linearly ordered set H is sequentially compact and its density is equal to $\mathbf{s}$.*

Proof. Suppose that $S = ((x_n, y_n))_{n\in\mathbb{N}}$ is a monotone sequence of points in H. Then $(x_n)_{n\in\mathbb{N}}$ is monotone as well. If $(x_n)_{n\in\mathbb{N}}$ is eventually constant, say $x_n = x$ for all $n > m$, then $(y_n)_{n>m}$ is a monotone sequence. Hence $\lim_{n\to\infty} y_n$ exists and S converges to $(x, \lim_{n\to\infty} y_n)$. Otherwise there are infinitely many distinct x_n's. If S is increasing, then S converges to $(\lim_{n\to\infty} x_n, 0)$. If S is decreasing, then S converges to $(\lim_{n\to\infty} x_n, 1)$ provided that $\lim_{n\to\infty} x_n \in \mathcal{D}(\mathcal{S})$; otherwise it converges to $(\lim_{n\to\infty} x_n, 0)$.

Let Q be the set of rational numbers in the unit interval $[0, 1]$. Then

$$H_0 = (\mathcal{D}(\mathcal{S}) \times Q) \cup (Q \times \{0\})$$

is dense in H and of cardinality $\mathbf{s}$. This shows that $d(H) \leq \mathbf{s}$. If $H' \subseteq H$ is of cardinality less than $\mathbf{s}$, then there is some $s \in \mathcal{D}(\mathcal{S})$ such that $\{s\} \times [0, 1]$ is disjoint from H'. Hence H' is not dense in H. Thus we also have $d(H) \geq \mathbf{s}$. □

The following theorem is a variation of an example in [6].

Theorem 9. *There exists a sequence $(f_n)_{n\in\mathbb{N}}$ of increasing functions from the subset $\mathcal{D}(\mathcal{S})$ of the unit interval to the linearly ordered set H such that $(f_n)_{n\in\mathbb{N}}$ does not have any pointwise convergent subsequence.*

Proof. For each $n \in \mathbb{N}$ and $V \in \mathcal{S}$, let

$$f_n(\mathcal{D}(V)) = \begin{cases} (\mathcal{D}(V), 1), & \text{if } n \in V; \\ (\mathcal{D}(V), 0), & \text{otherwise.} \end{cases}$$

Each function f_n is obviously increasing. For any infinite subsequence $(f_n)_{n\in I}$, let $V \in \mathcal{S}$ be such that the sets $I \cap V$ and $I \setminus V$ are both infinite. Then we have $f_n(\mathcal{D}(V)) = (\mathcal{D}(V), 1)$ for every $n \in I \cap V$, and $f_n(\mathcal{D}(V)) = (\mathcal{D}(V), 0)$ for every $n \in I\setminus V$. In particular, the sequence of points $(f_n(\mathcal{D}(V)))_{n\in I}$ is not convergent. □

The theorem above shows that the condition $d(Y) < \mathbf{s}$ in Theorem 7 is optimal. Using this fact, we obtain the following characterization of the splitting number.

Let τ_1 be the supremum of the cardinals κ with the property that for every set of X of cardinality less than κ and for every sequentially compact linearly ordered set Y of density less than κ, any sequence of functions from X to Y has a pointwise convergent subsequence. Likewise, let τ_2 be the least cardinal κ such that, for some set X of cardinality κ, it is not the case that any sequence of functions from X to $\{0, 1\}$ has a pointwise convergent subsequence, where we consider $\{0, 1\}$ as a linearly ordered set with $0 < 1$. Finally, let μ be the supremum of the cardinals κ with the property that, for any linearly ordered set X and any sequentially compact linearly ordered set Y with $d(Y) < \kappa$, any sequence of monotone functions from X to Y has a pointwise convergent subsequence.

Theorem 10. $\mathbf{s} = \tau_1 = \tau_2 = \mu$.

Proof. By definition we have $\tau_1 \leq \tau_2$. Lemma 4 implies $\mathbf{s} \leq \tau_1$; $\mathbf{s} \leq \mu$ follows from Theorem 7. Theorem 9 implies $\mathbf{s} \geq \mu$.

To see $\tau_2 \leq \mathbf{s}$, we use a variant of Rademacher's functions (see [4]): for each $n \in \mathbb{N}$, the function φ_n on the family of all infinite subsets of $\mathbb{N}$ to $\{0, 1\}$ is defined by

$$\varphi_n(V) = \begin{cases} 1, & \text{if } n \in V; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{S}$ be a splitting family of cardinality $\mathbf{s}$. For any infinite subset $I \subseteq \mathbb{N}$, let $V \in \mathcal{S}$ be such that the sets $I \cap V$ and $I \setminus V$ are both infinite. Then the 0-1 sequence $(\varphi_n(V))_{n\in I}$ is not convergent as 0 and 1 both appear infinitely many times in this sequence. This shows that no subsequence of the sequence $(\varphi_n \restriction \mathcal{S})_{n\in\mathbb{N}}$ of functions from $\mathcal{S}$ to $\{0,1\}$ can be pointwise convergent. □

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