MIGRATION OF ZEROS FOR SUCCESSIVE DERIVATIVES
OF ENTIRE FUNCTIONS

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Abstract. It is shown that if \( f \) is an entire function of order less than one, all of whose zeros are real, then the minimal root of \( f^{(k)} \) is an increasing function of \( k \) which accelerates as \( k \) increases.

In his survey article on successive derivatives of analytic functions, G. Polya begins with the question, “How do the zeros of the \( n^{th} \) derivative \( f^{(n)}(x) \) behave when \( n \) becomes very large?” ([P]). He asks if one can find “some definite trend” in this “migration of zeros”. In that same paper, Polya describes his rather pleasing answer when \( f \) is a meromorphic function with at least one pole, in which case a complete description of the final set of \( f \) is provided.

There are many interesting results in the case where \( f \) is an entire function—generally speaking, as is pointed out in [P], “If the order of \( f(z) \) is less than one, the differentiation tends to scatter the zeros; the zeros of \( f^{(n)}(z) \) tend to move out to \( \infty \) as \( n \) increases. If the order of \( f(z) \) is greater than one, their distribution becomes denser” (see [B] for a survey of some results along these lines). In this article, we study the question in the setting where \( f \) is entire, of order less than one, and possessing only finitely many non-real zeros. Then, we know from [CCS] that for large \( n \), the zeros of \( f^{(n)}(z) \) are all real. We show that once this happens, the zeros start to scatter towards infinity in such a way that the zero free region grows in an accelerating fashion.

Let \( \mathcal{F} \) be the set of entire functions \( f \) of one variable satisfying the following two conditions: All the roots of \( f \) are real, and the order of \( f \) is less than one. One easily sees that \( f \in \mathcal{F} \) implies \( f' \in \mathcal{F} \). We shall study the migration properties of the zeros of \( f^{(k)} \) as a function of \( k \). Our main result says that the minimal root of \( f^{(k)} \) accelerates as \( k \) increases. To be precise, let

\[ r_k = \inf \{ x \in \mathbb{R} : f^{(k)}(x) = 0 \} \]

for all \( k \geq 0 \) such that \( f^{(k)} \neq 0 \).

**Theorem 1.** The sequence \( \{r_k\} \) has non-negative velocity and non-negative acceleration. That is, if we let \( v_k = r_{k+1} - r_k \) and \( a_k = v_{k+1} - v_k \), then \( v_k \geq 0 \) and \( a_k \geq 0 \) for \( k \geq 0 \).
Remark 1. It is natural to ask what happens if we relax the condition that \( f \) be of order less than one, and merely impose the condition that \( f \) and all its derivatives have no non-real zeros. This class of functions has been identified with the class of Laguerre-Pólya functions ([HW1], [HW2]); such functions all have order less than two. It is easy to find functions in this class which violate the conclusion of Theorem 1. For example, if we let \( f(z) = e^{-z}(z^2 - az) \) where \( a \in \mathbb{R} \) and \( a < 0 \), then \( r_k = \min(0, a + 2k) \).

Remark 2. It follows from Theorem 1 that \( \lim_{k \to \infty} r_k = \infty \) when \( f \) has a smallest zero. This fact is known and appears in [W] (see also [G], page 35).

Before proving the theorem, we first verify that \( \mathcal{F} \) is closed under differentiation. Recall that the order of a function \( f \) is the smallest \( \lambda = \lambda(f) \geq 0 \) with the following property: For every \( \epsilon > 0 \) there exists \( C > 0 \) such that
\[
\sup \{|f(z)| : |z| = r\} < C e^{\lambda + \epsilon}.
\]

The Cauchy integral formula implies that \( \lambda(f) \geq \lambda(f') \). Thus, to see that \( \mathcal{F} \) is closed under differentiation, it suffices to observe that \( f \in \mathcal{F} \) implies that all the roots of \( f' \) are real. Now \( \lambda(f) < 1 \) implies that \( f \) has a product expansion which converges uniformly on compact subsets of \( \mathbb{C} \):
\[
f(z) = K z^m \prod_{n} \left(1 - \frac{z}{a_n} \right)
\]
where \( K \in \mathbb{C} \), \( m \) is a non-negative integer, and the \( a_n \) range over the non-zero roots of \( f \) (counted with multiplicity) ([A], Theorem 8, chapter 5). Thus \( f \) can be uniformly approximated on compact sets by polynomials with real zeros; since the set of such polynomials is closed under differentiation, we see that \( \mathcal{F} \) is also closed under differentiation.

Next we observe that to prove the theorem, it suffices to consider the case where \( f \) is a polynomial. First we note that if \( r_0 = -\infty \), then \( r_k = -\infty \) for all \( k \), in which case there is nothing to prove. Thus in (1) we may assume that \( r_0 = a_0 \) and that \( a_n \leq a_{n+1} \) for all \( n \). Let \( f_N \) be defined by (1) with the product restricted to those \( n \leq N \). Then \( f_N \) converges to \( f \) uniformly on compact subsets, and thus it suffices to prove the theorem for each \( f_N \).

Henceforth we shall assume that \( f \) is a polynomial of degree \( m \). It is clear that \( r_k \) has non-negative velocity. Thus we are reduced to showing that the acceleration is positive. We may assume \( m \geq 3 \) (otherwise there is nothing to prove). We change notation slightly, and rephrase the main theorem: Let \( f(x) \) be a polynomial of degree \( m \) with real coefficients. Assume \( m \geq 3 \) and that all the roots of \( f \) are real. Let \( n = m - 1 \) and let \( C \) be the “center” of \( f \), that is, \( C \) is the average of the the roots of \( f \). For \( 0 \leq k \leq n \) define
\[
R_k = \max \{ x \in \mathbb{R} : f^{(k)}(x) = 0 \},
\]
\[
r_k = \min \{ x \in \mathbb{R} : f^{(k)}(x) = 0 \}.
\]

It is clear that \( r_0 \leq r_1 \leq \ldots \leq r_n = C = R_n \leq R_{n-1} \leq \ldots R_0 \).

**Theorem 2.** The sequences \( \{r_k\} \) and \( \{R_k\} \) accelerate towards the center. That is,
\[
r_k - r_{k-1} \leq r_{k+1} - r_k \quad \text{and} \quad R_{k-1} - R_k \leq R_k - R_{k+1}
\]
for \( 1 \leq k \leq n - 1 \).
Lemma 1. Let $n \geq 2$ and let $\alpha_1, \ldots, \alpha_n$ be a decreasing sequence of real numbers. Let
\[ g(x) = \prod_{i=1}^{n} (x - \alpha_i) . \]
Let $\beta$ be the largest root of $g'(x) = 0$. Then
\[ \beta \geq \frac{(\alpha_1 + \alpha_2)}{2} . \]
Proof. If $\alpha_1 = \alpha_2$, then $\beta = \alpha_1 = \alpha_2$, and the result is clear. Thus we may assume that $\alpha_1 > \beta > \alpha_2$. Then
\[ 0 = \frac{g'(\beta)}{g(\beta)} = \sum_{i=1}^{n} \frac{1}{\beta - \alpha_i} \geq \frac{1}{\beta - \alpha_1} + \frac{1}{\beta - \alpha_2} , \]
which implies the result.

We now return to the proof of the theorem. First we note that if $a, b, c \in \mathbb{R}$ with $ac \neq 0$, then it suffices to prove the theorem for $cf(ax + b)$. In fact, replacing $f(x)$ by $f(-x)$, we are reduced to proving that $\{R_k\}$ accelerates.

By induction on the degree of $f$, we need only show that $R_0 - R_1 \leq R_1 - R_2$. Replacing $f(x)$ by $f(ax + b)$ for appropriate $a, b \in \mathbb{R}$, we may assume that $R_0 = 2$ and $R_2 = 0$. Thus, our task is to prove $R_1 \geq 1$. If $f$ has two or more non-negative roots, then the lemma implies that $R_1 \geq 1$. Thus we may assume that $f$ can be written in the form:

\[ f(x) = (x - 2) \prod_{i=1}^{n} (1 + \rho_i x) , \]

with $\rho_i > 0$.

Let
\[ G(\vec{\rho}) = G(\rho_1, \ldots, \rho_n) = \sum_{i=1}^{n} \rho_i - 2 \sum_{i<j} \rho_i \rho_j . \]

Then $R_2 = 0$ implies $G(\vec{\rho}) = 0$. To show $R_1 \geq 1$ it suffices to prove that $f'(1) \leq 0$. Since $f(1) < 0$, we are reduced to proving $f'(1)/f(1) \geq 0$, that is, if we define
\[ F(\vec{\rho}) = \sum_{i=1}^{n} \frac{\rho_i}{1 + \rho_i} , \]
then we must show that $G(\vec{\rho}) = 0$ implies $F(\vec{\rho}) \geq 1$.

Let
\[ S = \{ \vec{\rho} \in \mathbb{R}^n : G(\vec{\rho}) = 0, \rho_i \geq 0 \text{ for all } i \text{ and } \rho_i > 0 \text{ for some } i \} . \]

Lemma 2. Let $0 < \epsilon < 1/2(n - 1)$.
If $\vec{\rho} \in S$, then $\rho_i > \epsilon$ for some $i$.
If $\vec{\rho} \in S$ and $\rho_i > 0$ for some $i$, then $\rho_j > \epsilon$ for some $j \neq i$. 
Proof. If $\rho_i < 1/(n-1)$ for all $i$, and if $\rho_1 > 0$ for some $i$, then

$$G(\vec{\rho}) = \sum_{i=1}^{n} \rho_i (1 - \sum_{j \neq i} \rho_j) > 0.$$  

Now assume that $\rho_1 > 0$, and assume that $\rho_i < 1/(n-1)$ for all $i > 1$. Then we have:

$$G(\vec{\rho}) = \rho_1 (1 - 2 \sum_{i>1} \rho_i) + \sum_{i>1} \rho_i (1 - \sum_{j \neq i} \rho_j) > 0.$$  

Lemma 3. The function $F$ achieves its minimum on the set $S$.

Proof. Note that if $\rho_i = 1/(n-1)$ for all $i$, then $\vec{\rho} \in S$ and $F(\vec{\rho}) = 1$. Now choose $K$ to be a large real number, and let $S(K) = \{ \vec{\rho} \in S : \rho_i \leq K \text{ for all } i \}$.

Then, by Lemma 2, the set $S(K)$ is compact. Also, if $\vec{\rho} \in S$ and $\vec{\rho} / \in S(K)$, then

$$F(\vec{\rho}) > \frac{K}{K+1} + \frac{\epsilon}{\epsilon+1} > 1$$

for $K$ sufficiently large (by Lemma 2). This proves Lemma 3.

Let $S_{\text{min}}$ be the set of points where $F$ achieves its minimum value. Let $T \subseteq S_{\text{min}}$ be the subset consisting of those points with the maximal number of components which are zero.

Lemma 4. Let $\vec{a} \in T$ be a point with a minimum number of distinct non-zero entries. Then all the non-zero entries of $\vec{a}$ are equal and $F(\vec{a}) = 1$.

Proof. Assume not: Then we may assume $a_1 \neq a_2$ and $a_1 a_2 \neq 0$. Let $A = a_3 + \cdots + a_n$ and $B = 2 \sum_{3 \leq i < j} a_i a_j$, and let $x = \rho_1 + \rho_2$ and $y = 2 \rho_1 \rho_2$. Consider the function

$$g(x, y) = G(\rho_1, \rho_2, a_3, ..., a_n) = x(1 - 2A) - y + A - B.$$  

The domain of $g$ is

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 \geq 2y \geq 0 \}.$$  

Let

$$f(x, y) = F(\rho_1, \rho_2, a_3, ..., a_n) = \frac{\rho_1}{1+\rho_1} + \frac{\rho_2}{1+\rho_2} + C = \frac{x+y}{1+x+y/2} + C$$

where $C$ is a constant depending only on $a_3, ..., a_n$. Now our assumptions imply that if we restrict $f(x, y)$ to the line segment $g(x, y) = 0, (x, y) \in D$, then the minimum of $f$ occurs in the interior of the line segment. But the function $f(x, x(1-2A)+A-B)$ is a non-constant linear fractional transformation, and therefore has no local minima (or maxima), a contradiction. Thus all the non-zero entries of $\vec{a}$ are equal, which yields $F(\vec{a}) = 1$. This proves the lemma, and hence the theorem.
REFERENCES


