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SMALL VALUES OF POLYNOMIALS AND POTENTIALS WITH L_p NORMALIZATION

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Abstract. For a polynomial P of degree $\leq n$, normalized by the condition

$$\frac{1}{2\pi}\int_0^{2\pi} \mid P(re^{i\theta}) \mid^p d\theta = 1,$$

we show that $E(P;r;\varepsilon) := \{z : \mid z \mid \leq r, \mid P(z) \mid \leq \varepsilon^n\}$ has *cap* at most $r\varepsilon\kappa_{np}$, where $\kappa_{np} \leq 2$ is explicitly given and sharp for each n, r. Similar estimates are given for other normalizations, such as $p = 0$, and for planar measure, and for generalized polynomials and potentials, thereby extending work of Cuyt, Driver and the author for $p = \infty$. The relation to Remez inequalities is briefly discussed.

1. Results and proofs

A famous lemma of Cartan estimates the size of the lemniscate $\{z : \mid P(z) \mid \leq \varepsilon^n\}$ (measured by α-dimensional content) when P is a monic polynomial of degree n. Of course, the logarithmic capacity (*cap*) of such a set is just ε. Using these types of results for monic polynomials, Pommerenke, for example, showed [9] that for any polynomial P of degree $\leq n$,

$$cap\Big\{z : \mid z \mid \leq r \text{ and } \parallel P \parallel_{L_\infty(|z|\leq r)} / \mid P(z) \mid \geq \varepsilon^n\Big\} \leq 3r\varepsilon.$$

This was an essential ingredient of the Nuttall-Pommerenke theorem for Padé approximants.

Recently, A. Cuyt, K. Driver, and the author [2] showed that a simpler approach involving the Green's function and Bernstein-Walsh inequalities allows one to replace $3r\varepsilon$ by $2r\varepsilon$ and that this is sharp for each n, r. This was then applied to other measures of size and multivariate polynomials. In this note, we obtain sharp estimates when sup norms are replaced by L_p ones. Our estimates apply to logarithmic capacity *cap* and planar Lebesgue measure *meas*. Using standard inequalities, one can deduce estimates for Hausdorff contents and linear measure [7, pp. 202–203], [5, p. 300]. We believe the simplicity of the approach, and the sharpness of the constants, justify some attention. (For further orientation, the reader may refer to [2] and the survey article [8].)

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Our main result deals not only with polynomials, but also with generalized polynomials, and more generally exponentials of potentials. Recall that a *generalized polynomial of degree at most* n is an expression

$$P(z)=\prod_{j=1}^{m}\mid z-z_j\mid^{\alpha_j}:\alpha_j\in(0,\infty);\quad \sum_{j=1}^{m}\alpha_j\le n.$$

The latter are in turn a special case of exponentials of potentials. Given a nonnegative Borel measure ω with compact support and total mass at most n, and $c\in[0,\infty)$, we say that

$$P(z)=c\exp\left(\int\log\mid z-t\mid d\omega(t)\right)$$

is an *exponential of a potential of degree at most* n. Note that if ω consists of finitely many point masses, then P is a generalized polynomial of degree at most n. See [1], [4] for further orientation on generalized polynomials.

Let us define for $r>0$

$$\| P \|_{L_p(|z|=r)}:=\begin{cases}\text{ess sup}_{\theta\in[-\pi,\pi]}\left|P(re^{i\theta})\right|, & p=\infty,\\ \left(\frac{1}{2\pi}\int_0^{2\pi}\mid P(re^{i\theta})\mid^p d\theta\right)^{1/p}, & 0<p<\infty,\\ \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\mid P(re^{i\theta})\mid d\theta\right), & p=0.\end{cases}$$

The L_0 norm of a polynomial is sometimes called its Mahler measure [1].

Theorem 1.1. *Let* r, $\varepsilon>0$ *and* $0\le p\le\infty$. *Let* P *be an exponential of a potential of degree at most* n, *normalized by the condition*

$$\| P \|_{L_p(|z|=r)}=1. \tag{1.1}$$

Let

$$E(P;r;\varepsilon):=\left\{z:\mid z\mid\le r,\mid P(z)\mid\le\varepsilon^n\right\}. \tag{1.2}$$

Set $\kappa_0:=1$; $\kappa_\infty=2$ *and*

$$\kappa_\lambda:=2\left[\frac{\Gamma(\frac{\lambda+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{\lambda}{2}+1)}\right]^{1/\lambda}\ (\le 2),\quad 0<\lambda<\infty. \tag{1.3}$$

Then

$$cap\big(E(P;r;\varepsilon)\big)\le r\varepsilon\kappa_{np};\ meas\big(E(P;r;\varepsilon)\big)\le\pi(r\varepsilon\kappa_{np})^2. \tag{1.4}$$

These are sharp in the sense that for fixed n,r,

(1.5)
$$\sup_{\substack{\varepsilon>0\\ \deg(P)=n}}\frac{cap\big(E(P;r;\varepsilon)\big)}{\varepsilon}=r\kappa_{np};\quad \sup_{\substack{\varepsilon>0\\ \deg(P)=n}}\frac{meas\big(E(P;r;\varepsilon)\big)}{\varepsilon^2}=\pi(r\kappa_{np})^2.$$

Here the sup *is taken over ordinary polynomials* P *of degree* n.

We note that there is continuity as $p\to\infty$, since

$$\kappa_\lambda=2+O\left(\frac{\log\lambda}{\lambda}\right),\ \lambda\to\infty.$$

Of course, the restriction that n be an integer above can be dropped, without any changes to the proof of (1.4). For (1.5), one can then no longer consider polynomials, but generalized polynomials.

We shall also prove a generalization of Theorem 1.1:

Theorem 1.2. *Let r, p, $\varepsilon > 0$ and $\sigma \in \mathbb{R}$. Let $\psi : (0, \infty) \to \mathbb{R}$ be continuous and monotone increasing with*

$$\psi(0) := \lim_{t \to 0+} \psi(t) < \sigma < \lim_{t \to \infty} \psi(t) =: \psi(\infty).$$

Assume, moreover, that $\psi(e^t)$ is convex in $(-\infty, \infty)$. Let P be an exponential of a potential of degree at most n, normalized by the condition

$$\frac{1}{2\pi} \int_0^{2\pi} \psi\left(| P(re^{i\theta}) |^p\right) d\theta = \sigma. \tag{1.6}$$

For $\lambda > 0$, let $\kappa_{\lambda,\sigma,\psi}$ be the largest root of the equation

$$\frac{1}{2\pi} \int_0^{2\pi} \psi\left(\left[\frac{| 1 - e^{i\theta} |}{\kappa_{\lambda,\sigma,\psi}}\right]^\lambda\right) d\theta = \sigma. \tag{1.7}$$

Then

$$cap\big(E(P; r; \varepsilon)\big) \le r\varepsilon\kappa_{np,\sigma,\psi};\ meas\big(E(P; r; \varepsilon)\big) \le \pi(r\varepsilon\kappa_{np,\sigma,\psi})^2. \tag{1.8}$$

If ψ is in addition strictly increasing, these are sharp in the sense that for fixed n, r,

(1.9)

$$\sup_{\substack{\varepsilon>0 \\ \deg(P)=n}} \frac{cap(E\big(P; r; \varepsilon)\big)}{\varepsilon} = r\kappa_{np,\sigma,\psi};\quad \sup_{\substack{\varepsilon>0 \\ \deg(P)=n}} \frac{meas(E\big(P; r; \varepsilon)\big)}{\varepsilon^2} = \pi(r\kappa_{np,\sigma,\psi})^2.$$

Here the sup *is taken over all ordinary polynomials P of degree n.*

We note one further extension:

Theorem 1.3. *Let r, p, ε, ψ, σ, P be as in Theorem 1.2 with the additional condition that ψ has a finite limit at 0. Let ν be a positive Borel measure with compact support. Let P be normalized by the condition*

$$\int \psi\left(| P(z) |^p\right) d\nu(z) = \sigma. \tag{1.10}$$

Let $\kappa_{\lambda,\sigma,\psi,\nu}$ be the largest root of the equation

$$\max_{|t| \le r} \int \psi\left(\left[\frac{| z - t |}{\kappa_{\lambda,\sigma,\psi,\nu}}\right]^\lambda\right) d\nu(z) = \sigma. \tag{1.11}$$

Then

(1.12)

$$cap\big(E(P; r; \varepsilon)\big) \le r\varepsilon\kappa_{np,\sigma,\psi,\nu};\ meas\big(E(P; r; \varepsilon)\big) \le \pi(r\varepsilon\kappa_{np,\sigma,\psi,\nu})^2$$

and if ψ is strictly increasing, these are sharp in the sense that for fixed n, r,

(1.13)

$$\sup_{\substack{\varepsilon>0 \\ \deg(P)=n}} \frac{cap\big(E(P; r; \varepsilon)\big)}{\varepsilon} = r\kappa_{np,\sigma,\psi,\nu};\quad \sup_{\substack{\varepsilon>0 \\ \deg(P)=n}} \frac{meas_2\big(E(P; r; \varepsilon)\big)}{\varepsilon^2} = \pi(r\kappa_{np,\sigma,\psi,\nu})^2.$$

We note that one may extend the result still further to the weighted capacities treated by Saff and Totik [11]. However, the main application seems to be in rational approximation, so we omit these.

The inequalities above have some relation to Remez inequalities. Indeed at first sight, the following corollary seems to be of Remez type for the unit ball.

Corollary 1.4. *Let* r, $\varepsilon > 0$ *and* $0 \leq p \leq \infty$. *Let* P *be an exponential of a potential of degree* $\leq n$. *Then*

$$\| P \|_{L_p(|z|=r)} \leq \left(\frac{r\varepsilon\kappa_{np}}{cap\big(E(P;r;\varepsilon)\big)} \right)^n ; \tag{1.14}$$

$$\| P \|_{L_p(|z|=r)} \leq \left(\frac{\sqrt{\pi} r\varepsilon\kappa_{np}}{\sqrt{meas\big(E(P;r;\varepsilon)\big)}} \right)^n . \tag{1.15}$$

These are sharp in the sense that for each r, n, *we can find an ordinary polynomial* P *of degree* n *and* $\varepsilon > 0$ *for which both sides of (1.14), (1.15) are arbitrarily close to* 1.

The apparent similarity to Remez inequalities (which are deeper) is misleading. The above is sharp as $meas\big(E(P;r;\varepsilon)\big)$ approaches 0, whereas the rationale and utility of Remez inequalities is usually for the case where it is as large as possible, namely when it approaches πr^2. See for example Theorem 2.5 of [4], which applies to the unit ball when $meas\big(E(P;r;\varepsilon)\big)$ is bounded away from 0. In turn, some of the Remez inequalities imply cruder forms of Theorem 1.1, 1.2. See [8] for a more detailed comparison. We shall prove Theorem 1.2 and deduce Theorem 1.1 and Corollary 1.4. The reader will easily see how to similarly prove Theorem 1.3.

Proof of Theorem 1.2. We let $E := E(P;r;\varepsilon)$, a bounded G_δ set (as P is upper semi-continuous). If $cap(E) = 0$, the first inequality in (1.8) is trivial, so we assume that the capacity is positive. We may also assume that E is compact. (Indeed if E is not compact, then the argument below applies to any compact subset of E, and the inner regularity of logarithmic capacity and Lebesgue measure then give (1.8) in general.) As E is a compact set, it has a Green's function

$$g(z) := \int \log | z - t | \, d\mu(t) + \log \frac{1}{cap(E)}$$

so that μ is a unit positive measure with support in E, the so-called equilibrium measure of E. Note that $g \geq 0$ in $\mathbb{C}$ and $g = 0$ in E outside a set of $cap\, 0$ [12, pp. 224-228]. Then the Bernstein-Walsh inequality gives:

$$P(z) \leq e^{ng(z)} \| P \|_{L_\infty(E)} \leq e^{ng(z)} \varepsilon^n. \tag{1.16}$$

This is often stated only for absolute values of ordinary polynomials, and in the above form is really a special case of the second maximum principle for subharmonic functions. Let us sketch the proof: since $g \geq 0$, we may assume that P has degree n. The function

$$F(z) := \log P(z) - ng(z) - \log \| P \|_{L_\infty(E)}$$

is subharmonic in $\mathbb{C} \backslash E$ and ≤ 0 in E and has a finite limit at ∞. The fact that $g \geq 0$ and the upper semi-continuity of $\log P$ give for boundary points z_0 of E, and

z approaching z_0 from outside E,

$$\begin{aligned}\limsup_{z\to z_0} F(z) &\leq \limsup_{z\to z_0}\Big(\log P(z) - \log \| P \|_{L_\infty(E)}\Big)\\ &\leq \log P(z_0) - \log \| P \|_{L_\infty(E)} \leq 0.\end{aligned}$$

By the second maximum principle for subharmonic functions [12, p.223], $F \leq 0$ in each of the components of $\mathbb{C}\backslash E$. Thus $F \leq 0$ in $\mathbb{C}$ and we have (1.16). Next, by our normalization (1.6), and then by (1.16) and Jensen's inequality applied to the convex function $\psi(e^t)$, (recall that μ has total mass 1)

$$\begin{aligned}\sigma &= \frac{1}{2\pi}\int_0^{2\pi} \psi\left(| P(re^{i\theta}) |^p\right) d\theta\\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \psi\left(e^{\int np[\log|re^{i\theta}-t| + \log\frac{\varepsilon}{cap(E)}]d\mu(t)}\right) d\theta\\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\int \psi\left(\left[\frac{| re^{i\theta} - t | \varepsilon}{cap(E)}\right]^{np}\right) d\mu(t)d\theta\\ &= \int \frac{1}{2\pi}\int_0^{2\pi} \psi\left(\left[\frac{| re^{i\theta} - t | \varepsilon}{cap(E)}\right]^{np})d\theta\right) d\mu(t)\\ &\leq \sup_{|t|\leq r}\frac{1}{2\pi}\int_0^{2\pi} \psi\left(\left[\frac{| re^{i\theta} - t | \varepsilon}{cap(E)}\right]^{np}\right) d\theta\\ &= \sup_{\tau\in[0,1]}\frac{1}{2\pi}\int_0^{2\pi} \psi\left(\left[\frac{| 1 - \tau e^{i\theta} | r\varepsilon}{cap(E)}\right]^{np}\right) d\theta.\end{aligned}$$

In the third last line, we used Fubini's theorem, which is applicable as the integrand is bounded above. Now the function $z \to np\log | 1 - z | + np\log\frac{r\varepsilon}{cap(E)}$ is subharmonic in the plane. Moreover, $t \to \psi(e^t)$ is convex, and composition of a subharmonic function with a convex one preserves subharmonicity, so $\psi\left(\left[\frac{|1-z|r\varepsilon}{cap(E)}\right]^{np}\right)$ is subharmonic and continuous for $| z |< 1$. It follows that the last integral increases with τ (see [10, pp. 336-337]) and hence we may continue this as

$$\sigma \leq \frac{1}{2\pi}\int_0^{2\pi} \psi\left(\left[\frac{| 1 - e^{i\theta} | r\varepsilon}{cap(E)}\right]^{np}\right) d\theta.$$

(Monotone convergence allows us to let $\tau \to 1$ and deduce convergence of the integral for $\tau = 1$: consider separately $\psi(0)$ finite or $-\infty$.) Now for fixed $\lambda > 0$, the function

$$\kappa \to \frac{1}{2\pi}\int_0^{2\pi} \psi\left(\left[\frac{| 1 - e^{i\theta} |}{\kappa}\right]^{\lambda}\right) d\theta$$

is monotone decreasing in κ, and has limit $\psi(\infty) > \sigma$ as $\kappa \to 0+$ and limit $\psi(0) < \sigma$ as $\kappa \to \infty$. Moreover, it is a continuous function of $\kappa > 0$. (To see this, note that if we omit small neighbourhoods of $\theta = 0, 2\pi$, then the resulting integral is continuous in κ. Moreover, the monotonicity of ψ shows that the tail integrals for θ close to $0, 2\pi$ may be bounded above for $\kappa \geq \kappa_0$ by the corresponding tail integrals for $\kappa = \kappa_0$. The mere convergence of the whole integral for $\kappa = \kappa_0$ allows one to estimate the tail integrals.) Thus (1.7) has a root $\kappa_{\lambda,\sigma,\psi}$ and by continuity, this may be taken as the largest root. Then the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\psi\left(\left[\frac{|\,1-e^{i\theta}\,|}{\kappa_{np,\sigma,\psi}}\right]^{np}\right)d\theta=\sigma\leq\frac{1}{2\pi}\int_0^{2\pi}\psi\left(\left[\frac{|\,1-e^{i\theta}\,|\,r\varepsilon}{cap(E)}\right]^{np}\right)d\theta$$

forces

$$\kappa_{np,\sigma,\psi}\geq\frac{cap(E)}{r\varepsilon}$$

and then the first inequality in (1.8) follows. Polya's inequality [5, p. 300]

$$meas_2(E)\leq\pi\big(cap(E)\big)^2$$

then gives the second inequality in (1.8).

We turn to the proof of the sharpness. Let $0<a<r$ and

$$P(z):=\big(c(z-a)\big)^n$$

where $c>0$ is chosen to give P with normalization (1.6), so that

$$\sigma=\frac{1}{2\pi}\int_0^{2\pi}\psi\Big(\big[c\,|\,re^{i\theta}-a\,|\,\big]^{np}\Big)d\theta.$$

Our extra hypothesis that ψ is strictly increasing shows that if $a\to r$, then $cr\to 1/\kappa_{np,\sigma,\psi}$. If ε is small enough, then the ball centre a, radius ε/c, is contained in the ball $|\,z\,|\leq r$. Then we see that for such ε,

$$E(P;r;\varepsilon)=\big\{z:|\,z-a\,|\leq\varepsilon/c\big\}$$

and hence

$$cap\big(E(P;r;\varepsilon)\big)=\varepsilon/c;\quad meas\big(E(P;r;\varepsilon)\big)=\pi(\varepsilon/c)^2.$$

The fact that $1/c$ may be made arbitrarily close to $r\kappa_{np,\sigma,\psi}$ establishes (1.9). □

For $p=\infty$, the proof of Theorem 1.1 is easy and exactly as in [2], so is omitted. We turn to the more difficult $p\in[0,\infty)$, applying Theorem 1.2 for suitable σ,ψ.

Proof of Theorem 1.1 for $0<p<\infty$. We evaluate $\kappa_{\lambda,\sigma,\psi}$ for $\psi(t)=t$ and $\sigma=1$. Dropping the subscripts σ,ψ, we see that (1.7) gives

$$\begin{aligned}\kappa_\lambda^\lambda &= \frac{1}{2\pi}\int_0^{2\pi}|\,1-e^{i\theta}\,|^\lambda\,d\theta=\frac{2^{\lambda+1}}{\pi}\int_0^{\pi/2}|\sin u\,|^\lambda\,du\\ &= \frac{2^\lambda}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda}{2}+1\right)}.\end{aligned}$$

See, for example, [3, p. 217, no.858.515]. To bound κ_λ for $\lambda>0$, we note that

$$\begin{aligned}\frac{\pi^{1/2}\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda}{2}+1\right)}&=B\left(\frac{1}{2},\frac{\lambda+1}{2}\right)=\int_0^1 t^{-1/2}(1-t)^{\lambda/2-1/2}dt\\ &\leq\int_0^1 t^{-1/2}(1-t)^{-1/2}dt=B\left(\frac{1}{2},\frac{1}{2}\right)=\pi.\end{aligned}$$

Thus,

$$\kappa_\lambda\leq 2.$$

□

Proof of Theorem 1.1 for $p = 0$. We evaluate $\kappa_{\lambda,\sigma,\psi}$ for $\psi(t) = \log t$ and $\sigma = 0$. (It is easy to check that the hypotheses of Theorem 1.2 are fulfilled with $p = 1$ there.) Note also that

$$\| P \|_{L_0(|z|=r)} = 1 \Leftrightarrow \frac{1}{2\pi}\int_0^{2\pi} \psi\left(| P(re^{i\theta}) |\right) d\theta = 0.$$

From (1.7), dropping the subscripts σ, ψ,

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi} \psi\left(\left[\frac{| 1 - e^{i\theta} |}{\kappa_\lambda}\right]^\lambda\right) d\theta \\ &\Leftrightarrow \log\frac{1}{\kappa_\lambda} + \frac{1}{2\pi}\int_0^{2\pi} \log | 1 - e^{i\theta} | \, d\theta = 0. \end{aligned}$$

As the integral is 0 [10, p. 307], we obtain $\kappa_\lambda = 1$. □

We remark that the asymptotic for κ_λ given after Theorem 1.1 follows directly from Stirling's formula. We also note that $\psi(t) := \log^+ t := \max\{0, \log t\}$ satisfies the hypotheses for the first part of Theorem 1.2. Thus one may normalize exponentials of potentials P to have Nevanlinna "norm"

$$\| P \|_* := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log^+ | P(re^{i\theta}) | \, d\theta\right) = \exp(\sigma) > 1$$

and deduce bounds on the size of $cap\big(E(P; r; \varepsilon)\big)$. Finally, we give the

Proof of Corollary 1.4. Given an exponential of a polynomial P of degree at most n, let

$$\rho := \| P \|_{L_p(|z|=r)}$$

and $Q(z) := P(z)/\rho$. Then Q admits the normalization (1.1) and

$$E(P; r; \varepsilon) = E\left(Q; r; \frac{\varepsilon}{\rho^{1/n}}\right).$$

Theorem 1.1, applied to Q, gives

$$cap\big(E(P; r; \varepsilon)\big) \le r\frac{\varepsilon}{\rho^{1/n}}\kappa_{np}.$$

Rearranging this gives (1.14). Similarly we obtain (1.15). Finally from Theorem 1.1, we can obtain P for which the left-hand side of (1.14), (1.15) is 1, while $cap\big(E(P; r; \varepsilon)\big)$ is arbitrarily close to $r\varepsilon\kappa_{np}$ and $meas\big(E(P; r; \varepsilon)\big)$ is arbitrarily close to $\pi(r\varepsilon\kappa_{np})^2$. □

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