EXTENSIONS OF PERFECT GO-SPACES
AND $\sigma$-DISCRETE DENSE SETS

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Abstract. In this paper, we prove that if a perfect GO-space $X$ has a $\sigma$-discrete dense set, then $X$ has a perfect linearly ordered extension. This answers a problem raised by H. R. Bennett, D. J. Lutzer and S. Purisch. And the result is also a partial answer to an old problem posed by H. R. Bennett and D. J. Lutzer.

1. Introduction

A GO-space (generalized ordered space) is a triple $\langle X, \tau, \leq \rangle$, where $\langle X, \leq \rangle$ is a linearly ordered set, $\tau$ a topology on $X$ which is $T_1$ and has the base consisting of open sets which are order-convex. If we denote the usual interval topology on $X$ by $\lambda$, then $\langle X, \lambda, \leq \rangle$ is called a LOTS (linearly ordered topological space). We say that $\langle X, \lambda, \leq \rangle$ is an underlying LOTS of the GO-space $\langle X, \tau, \leq \rangle$. If a GO-space $\langle X, \tau, \leq \rangle$ can be topologically embedded in a LOTS $\langle Y, \lambda, \prec \rangle$, then the LOTS $\langle Y, \lambda, \prec \rangle$ is called an orderable extension of the GO-space $\langle X, \tau, \leq \rangle$ and if $\leq = \prec | X$, then the LOTS $\langle Y, \lambda, \prec \rangle$ is called a linearly ordered extension of the GO-space $\langle X, \tau, \leq \rangle$.

It is an interesting question whether a topological property on a GO-space can be reflected on some of its orderable extensions. It is known that for separability, metrizability and paracompactness the answers to this question are affirmative (cf. [1]). But the following question posed by H. R. Bennett and D. J. Lutzer remains open.

Problem 1 ([1]). Is it true that any perfect GO-space has a perfect orderable extension?

In [6] and [7] the author with T. Miwa and Y.-Z. Gao has proved that there exists a perfect GO-space which cannot be densely embedded in any perfect LOTS. On the other hand any perfect GO-space with the underlying LOTS satisfying local perfectness can be embedded in a perfect LOTS. Recently H. R. Bennett, D. J. Lutzer and S. Purisch studied dense subspaces of GO-spaces; they posed the following question.

Problem 2 ([2]). Is it true that a GO-space with a $\sigma$-discrete dense subset can be embedded in a LOTS with a $\sigma$-discrete dense subset?
The aim of the present paper is to give a solution to Problem 2 in the affirmative and to point out relations between Problem 1 and an older set theoretic problem (see Section 3).

For a GO-space \( \langle X, \tau, \leq \rangle \), let
\[
\begin{align*}
\lambda &\quad \text{be the interval topology on } \langle X, \leq \rangle, \\
I &= \{ x \in X \mid \{ x \} \in \tau - \lambda \}, \\
R &= \{ x \in X - I \mid [x, \to) \in \tau - \lambda \}, \\
L &= \{ x \in X - I \mid (\to, x] \in \tau - \lambda \}, \\
E &= X - (R \cup L \cup I).
\end{align*}
\]

It is well-known that a GO-space topology on \( \langle X, \leq \rangle \) can be determined by the sets \( I, R, L, E \). So we denote the GO-space \( \langle X, \tau, \leq \rangle \) by \( GO_X(R, E, I, L) \) and write \( X = GO(R, E, I, L) \), simply saying \( X \) is a GO-space. By ‘discrete’ we always mean ‘closed discrete’.

2. Main results

To prove our results, we state a known result proved by the author.

**Theorem 1** ([5]). A perfect GO-space \( X = GO(R, E, I, L) \) has a perfect linearly ordered extension if and only if there exists a \( \sigma \)-discrete subset \( F \) of \( X \) such that \( X' = GO_X(\emptyset, X - F, F, \emptyset) \) is perfect.

**Lemma 2.** Let \( X = GO(R, E, I, L) \) be a GO-space and \( Y \) the underlying LOTS of \( X \). If \( D \) is a discrete subset of \( X \), then there exists a discrete subset \( D' \supseteq D \) of \( X \) such that \( D' \) is closed in \( Y \).

**Proof.** Let \( D' = cl_Y D \). It is sufficient to prove that \( D' \) is discrete in \( X \).

For \( x \in X \), if \( x \in I \), \( \{ x \} \) is an open neighborhood of \( x \) in \( X \) which intersects \( D \) in at most one point.

If \( x \in R \), there exists \( y > x \) such that \( [x, y] \cap D = \{ x \} \) or \( [x, y] \cap D = \emptyset \) since \( D \) is discrete in \( X \). If \( (x, y) \cap (D' - D) \neq \emptyset \), we would have \( (x, y) \cap D \neq \emptyset \). So \( (x, y) \cap D' = \{ x \} \) or \( (x, y) \cap D' = \emptyset \). Similarly if \( x \in L \), we may choose a \( y < x \) such that \( (y, x] \cap D' = \{ x \} \) or \( (y, x] \cap D' = \emptyset \).

If \( x \in E \), there exist \( y_0, y_1 \) with \( y_0 < x < y_1 \) such that \( (y_0, y_1) \cap D = \{ x \} \) or \( (y_0, y_1) \cap D = \emptyset \). If \( (y_0, y_1) \cap (D' - D) \neq \emptyset \), then \( |(y_0, y_1) \cap D| > 1 \). Therefore \( (y_0, y_1) \cap D' = \{ x \} \) or \( (y_0, y_1) \cap D' = \emptyset \). Hence \( D' \) is discrete in \( X \) and closed in \( Y \).

**Lemma 3** ([8]). If a GO-space \( X \) has a \( \sigma \)-discrete dense subset, then \( X \) is perfect.

**Theorem 4.** Let \( X = GO(R, E, I, L) \) be a perfect GO-space. If \( X \) has a \( \sigma \)-discrete dense subset \( F \), then \( X \) has a perfect linearly ordered extension.

**Proof.** Let \( Y \) be the underlying LOTS of \( X \). Since \( F \) is \( \sigma \)-discrete in \( X \), \( F = \bigcup \{ F_n \mid n \in \omega_0 \} \), where \( F_n \) is discrete in \( X \) for each \( n \in \omega_0 \). By Lemma 2, for each \( n \in \omega_0 \), we may choose a discrete subset \( F'_n \) of \( X \) such that \( F'_n \supseteq F_n \) and \( F'_n \) is closed in \( Y \).

Put \( F' = \bigcup \{ F'_n \mid n \in \omega_0 \} \). Then \( F' \) is an \( F_n \)-set in \( Y \) and a \( \sigma \)-discrete subset of \( X \). Consider the GO-space \( X' = GO_X(\emptyset, X - F', F', \emptyset) \). We prove that \( X' \) is perfect. It is obvious that \( F'_n \) is closed in \( X' \) for each \( n \in \omega_0 \). So \( F' \) is \( \sigma \)-discrete in \( X' \). Assume that \( x \in X - F' \) and \( (y_0, y_1) \) is a neighborhood of \( x \) in \( X' \). Since \( (y_0, y_1) \) is also open in \( X \) and \( F \subset F' \) is dense in \( X \), \( (y_0, y_1) \cap F' \neq \emptyset \). Thus \( F' \) is a
σ-discrete dense subset of $X'$. It follows from Lemma 3 that $X'$ is perfect. Hence by Theorem 1, $X$ has a perfect linearly ordered extension.

**Theorem 5.** If a GO-space $X = GO(R, E, I, L)$ has a σ-discrete dense subset, then $X$ has a perfect linearly extension with a σ-discrete dense subset.

**Proof.** By the proof of Theorem 4, it is known that if the GO-space $X$ has a σ-discrete dense subset, then $X$ satisfies the conditions of Theorem 1. With $R, L$ and $I$ as in Section 1, by the proof of Theorem 1 (see [5]), there exists a σ-discrete set $F$ of $X$ such that $I \subset F \subset R \cup L \cup I$, and the perfect linearly ordered extension of $X$ constructed in [5] has the form

$$P(X) = (X \times \{0\}) \cup ((R - F) \times \{-1\}) \cup ((L - F) \times \{1\})$$

$$\cup (I_0 \times (-1,1)) \cup ((I_- \cup (F \cap R)) \times (-1,0)) \cup ((I_+ \cup (F \cap L)) \times (0,1))$$

where

$$I_- = \{x \in I \mid \text{there is a } y \in X \text{ such that } x < y \text{ and } (x,y) = \emptyset\},$$

$$I_+ = \{x \in I \mid \text{there is a } y \in X \text{ such that } y < x \text{ and } (y,x) = \emptyset\},$$

$$I_0 = I - (I_- \cup I_+).$$

Since

$$O = \{\{x\} \times (-1,1) \mid x \in I_0\}$$

$$\cup \{\{x\} \times (-1,0) \mid x \in I_- \cup (F \cap R)\} \cup \{\{x\} \times (0,1) \mid x \in I_+ \cup (F \cap L)\}$$

is a σ-discrete collection in $P(X)$ and every element of the collection has a countable dense subset, there exists a σ-discrete subset of $P(X)$ which is dense in each element of $O$. For a point $\langle x, y \rangle \in P(X)$, if $x \notin F$, then the intersection of any neighborhood of $\langle x, y \rangle$ with $X \times \{0\}$ contains an interval of $X \times \{0\}$. It is easy to check that the σ-discrete subset of $X \times \{0\}$ is also σ-discrete in $P(X)$. Thus $P(X)$ has a σ-discrete dense subset.

**Remark**

By Theorem 4, we know that to find a counterexample to Problem 1, one must find a perfect GO-space which has no σ-discrete dense subset. But this is related to an old problem which is still open.

**Problem 3 ([1]).** Is there an example of a perfect GO-space in ZFC which does not have a σ-discrete dense subset?

So if the answer to Problem 3 is ‘no’, then there exists no counterexample in ZFC to Problem 1. On the other hand, it is well-known that if we assume that there exists a Souslin line $S$, the existence of which is independent of ZFC, then $S$ is a perfect LOTS which does not have a σ-discrete dense subset. However even under the assumption that Souslin line exists, any perfect GO-space with the Souslin line as the underlying LOTS does not serve as a counterexample to Problem 1 because by the result in [7], we know that any perfect GO-space with a perfect underlying LOTS has a perfect linearly extension.
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