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THE INTERVAL OF RESOLVENT-POSITIVITY FOR THE BIHARMONIC OPERATOR

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Abstract. For an operator A on a Banach lattice we examine the interval on the real line for which the resolvent $(\lambda - A)^{-1}$ is positive. This positivity interval is then explicitly calculated for the biharmonic operator $Af = -f''''$ with three different boundary conditions.

1. Introduction and formulation of the results

An important property of second-order elliptic operators is the maximum principle. It implies, in particular, that the realisation of such an operator in $C\left(\overline{\Omega}\right)$ or $C_0(\Omega)$ ($\Omega \subset \mathbb{R}^{\mathbb{N}}$ open) is resolvent positive (i.e. there exists a $\lambda_0 \in \mathbb{R}$ such that $(\lambda - A)^{-1}$ exists and $(\lambda - A)^{-1} \geq 0$ for all $\lambda > \lambda_0$). This property is never true for higher order differential operators (see e.g. [3] V 5.2, Theorem 1 for a precise formulation of this fact; see also Section 2). However, it can occur that the resolvent is positive on some compact interval. The purpose of this paper is to determine the precise interval of positivity for the bi-Laplacian on an interval with several boundary conditions. For the bi-Laplacian operator in higher dimensions, the interval of positivity can be void (see [2] or [6]).

The following operators are considered on the Banach space $E = C[0,1]$, or $E = L^p[0,1]$, $p \in [1,\infty]$.

Definition 1.1 (The physical operator)**.** The operator A_1 is defined by

$$D(A_1) = \{f \in E \mid f'''' \in E \text{ and } f(0) = f'(0) = f(1) = f'(1) = 0\},$$

and $A_1 f = -f''''$.

Definition 1.2 (The square operator)**.** The operator A_2 is defined by

$$D(A_2) = \{f \in E \mid f'''' \in E \text{ and } f(0) = f''(0) = f(1) = f''(1) = 0\},$$

and $A_2 f = -f''''$.

Definition 1.3 (One-sided boundary conditions)**.** The operator A_3 is defined by

$$D(A_3) = \{f \in E \mid f'''' \in E \text{ and } f(0) = f'(0) = f''(0) = f'''(0) = 0\},$$

and $A_3 f = -f''''$.

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The operator A_1 is called physical because it derives from a physical problem. For a flexible beam which is clamped at the ends at $x=0$ and $x=1$ and is subject to a force described by f, the beam bends to a shape u which satisfies the equation $A_1 u + f = 0$. The operator $-A_2$ is the square of the Laplace operator with Dirichlet boundary conditions on the unit interval.

In order to formulate our main result, we use the following notation. Let A be an operator on E and $\lambda \in \sigma(A)$ (the resolvent set of A). We write $(\lambda - A)^{-1} \geq 0$ if $0 \leq f \in E$ implies $(\lambda - A)^{-1} f \geq 0$. Here, for $f \in E$, we write $f \geq 0$ if $f(t) \geq 0$ for all $t \in [0,1]$, if $E = C[0,1]$ and $f(t) \geq 0$ a.e. if $E = L^p[0,1]$. In the examples considered here $(\lambda - A)^{-1}$ is given by an integral kernel $k(\lambda, x, y)$ (i.e. $(\lambda - A)^{-1} f(x) = \int_0^1 f(y) k(x, y, \lambda)\, dy$), and one has $(\lambda - A)^{-1} \geq 0$ if and only if $k(\lambda, x, y) \geq 0$ (x, y a.e.).

We are now in the position to formulate the main results of this paper

Theorem 1.1. *Let ζ_c be the first positive root of the equation $\tan z = \tanh z$, and let $\lambda_c = 4\zeta_c^4$ ($\zeta_c \approx 3.926602$ and $\lambda_c \approx 950.88427$). Then for $k = 1, 2, 3$*

$$(\lambda - A_k)^{-1} \geq 0 \text{ if } \lambda \in [0, \lambda_c]$$

and

$$(\lambda - A_k)^{-1} \not\geq 0 \text{ if } \lambda \in \left]\lambda_c, \infty\right[.$$

Using this theorem and the results of Section 3 we obtain

Theorem 1.2. *Let ζ_c be the first positive root of the equation $\tan z = \tanh z$ and let ζ_0 be the first positive root of the equation $\cos z \cosh z = 1$. Let $\lambda_c = 4\zeta_c^4$ and $\lambda_0 = -\zeta_0^4$ ($\lambda_0 \approx -500.563902$). Then the following hold.*

- *For the physical operator A_1, the interval of resolvent positivity is $]\lambda_0, \lambda_c]$.*
- *For the square operator A_2, the interval of resolvent positivity is $\left]-\pi^4, \lambda_c\right]$.*
- *For the operator with one-sided boundary conditions A_3, the interval of resolvent positivity is $]-\infty, \lambda_c]$.*

We will prove these theorems in the next sections. In Section 2 we will give some general results concerning the resolvent-positivity of an operator. In Section 3 the equations of the resolvents are solved to obtain the kernels of the operators A_k ($k = 1, 2, 3$). In Section 4 we will give conditions for the positivity of these kernels using inequalities which will be proved in Section 5. For the rest of this paper we will keep the definition of ζ_c given in the theorem.

2. General observations

A result by Arendt, Batty, and Robinson [1] (Proposition 2.2) states that if the resolvent $(\lambda - A)^{-1}$ of a linear differential operator with variable coefficients A is positive for all large λ, then the operator must be of order 2 (for the polyharmonic operator it is possible to show directly that the resolvent cannot remain positive, see [5]). However, for some operators it may be clear that the resolvent is positive at least at one point (e.g. for the operator $-A^2$ the resolvent is positive at zero when A is a resolvent-positive operator with negative spectral bound). It natural to ask how long the positivity remains.

We are now interested in the value λ_c where positivity will switch to non-positivity. We first note the following result.

Lemma 2.1. *Let A be a closed operator on a Banach lattice E such that $\lambda_0(A) := \sup \sigma(A) \cap \mathbb{R} < \infty$. Let $t > \lambda_0(A)$ be a real number such that $(t - A)^{-1} \geq 0$. Then for any $\tau \in \left]\lambda_0(A), t\right]$ also $(\tau - A)^{-1} \geq 0$.*

Proof. At each point $t \in \rho(A)$ we obtain

$$(\tau - A)^{-1} = \sum_{k=0}^{\infty} (t - \tau)^k \left((t - A)^{-1}\right)^{k+1},$$

which converges if $|\tau - t| < \operatorname{dist}(t, \sigma(A))$. □

From Lemma 2.1 we obtain immediately

Corollary 2.1. *For any closed operator A in a Banach lattice E, there exists a unique number $\lambda_c(A) \in [\lambda_0(A), \infty]$ such that $(\lambda - A)^{-1} \geq 0$ if $\lambda \in \left]\lambda_0(A), \lambda_c(A)\right]$ and $(\lambda - A) \ngeq 0$ for $\lambda \in \left]\lambda_c(A), \infty\right[$.*

Using this, the claim of Theorem 1.1 can be reformulated as

$$\lambda_c(A_1) = \lambda_c(A_2) = \lambda_c(A_3) = \lambda_c.$$

Furthermore, to prove Theorem 1.2 it has to be shown that

$$\lambda_0(A_1) = \lambda_0,$$

$$\lambda_0(A_2) = -\pi^4,$$

and

$$\lambda_0(A_3) = -\infty.$$

Greiner, Voigt, and Wolff gave in [4] (Example 3.10) this example of a matrix with finite λ_c. Define

$$B := \begin{pmatrix} 3 & 0 & 3 \\ 4 & 2 & 0 \\ 0 & 4 & 2 \end{pmatrix}$$

and let $A := -B^{-1}$. Then $\lambda_0(A) = -\frac{1}{6}$, $s(A) = -\frac{1}{20}$, and $\lambda_c(A) = 0$.

3. The kernels

To obtain the kernels of the operators A_k $(k = 1, 2, 3)$ we have to solve the equation

$$\lambda\phi + \phi'''' = f$$

with corresponding boundary conditions. The general solution of this equation is given by

$$\begin{aligned} \phi(x) \quad = \quad & \frac{1}{4\zeta^3} \int_0^x f(y) (\sin\zeta(x-y)\cosh\zeta(x-y) \\ & - \cos\zeta(x-y)\sinh\zeta(x-y))\, dy \\ & + c_0 \cos\zeta x \cosh\zeta x + c_1 \cos\zeta x \sinh\zeta x \\ & + c_2 \sin\zeta x \cosh\zeta x + c_3 \sin\zeta x \sinh\zeta x, \end{aligned} \tag{1}$$

where $\zeta = 4\lambda^{\frac{1}{4}}$ and the constants c_1, c_2, c_3, and c_4 have to be calculated from the boundary conditions.

We first examine the case of the one-sided boundary conditions $f(0) = f'(0) = f''(0) = f'''(0) = 0$. The equations for the constants in (1) are simply

$$c_0 = c_1 = c_2 = c_3 = 0,$$

and so the kernel k_3 for the operator A_3 is given by

$$k_3(x, y, \lambda) = \frac{1}{4\zeta^3} \left(\sin\zeta(x-y)\cosh\zeta(x-y) - \cos\zeta(x-y)\sinh\zeta(x-y) \right). \tag{2}$$

For the case of the square operator A_2 with boundary conditions $f(0) = f''(0) = f(1) = f''(1) = 0$ the equations are

$$c_0 = c_3 = 0,$$

$$\begin{aligned} & c_1 \cos\zeta \sinh\zeta + c_2 \sin\zeta \cosh\zeta \\ = \quad & \frac{1}{4\zeta^3} \int_0^1 f(y) \left(\cos\zeta(1-y)\sinh\zeta(1-y) - \sin\zeta(1-y)\cosh\zeta(1-y) \right) dy, \end{aligned}$$

and

$$\begin{aligned} & c_1 \sin\zeta \cosh\zeta - c_2 \cos\zeta \sinh\zeta \\ = \quad & \frac{1}{4\zeta^3} \int_0^1 f(y) \left(\cos\zeta(1-y)\sinh\zeta(1-y) + \sin\zeta(1-y)\cosh\zeta(1-y) \right) dy. \end{aligned}$$

Since the problem for the square operator is symmetric, we obtain $k_2(x, y, \lambda) = k_2(y, x, \lambda)$ and therefore we only have the calculate the kernel for $x \leq y$. Solving the equations leads to

$$\begin{aligned} k_2(x, y, \zeta) \quad = \quad & \frac{1}{4\zeta^3} \frac{1}{\sin^2\zeta + \sinh^2\zeta} \left(\sin\zeta\cosh\zeta \, \left(\sin\zeta x \cosh\zeta x \right. \right. \\ & \quad \times \left(\cos\zeta(1-y)\sinh\zeta(1-y) - \sin\zeta(1-y)\cosh\zeta(1-y) \right) \\ & + \cos\zeta x \sinh\zeta x \\ & \quad \left. \times \left(\cos\zeta(1-y)\sinh\zeta(1-y) + \sin\zeta(1-y)\cosh\zeta(1-y) \right) \right) \\ & + \cos\zeta\sinh\zeta \, \left(\sin\zeta x \cosh\zeta x \right. \\ & \quad \times \left(-\cos\zeta(1-y)\sinh\zeta(1-y) - \sin\zeta(1-y)\cosh\zeta(1-y) \right) \\ & + \cos\zeta x \sinh\zeta x \\ & \quad \left. \left. \times \left(\cos\zeta(1-y)\sinh\zeta(1-y) - \sin\zeta(1-y)\cosh\zeta(1-y) \right) \right) \right), \end{aligned} \tag{3}$$

for $0 \leq x \leq y \leq 1$.

Finally, for the case of the physical boundary conditions $f(0) = f'(0) = f(1) = f'(1) = 0$, we get the equations

$$c_0 = 0,$$

$$c_2 = -c_1,$$

$$\begin{aligned} & c_1 \left(\cos\zeta \sinh\zeta - \sin\zeta \cosh\zeta \right) + c_3 \sin\zeta \sinh\zeta \\ = \quad & \frac{1}{4\zeta^3} \int_0^1 f(y) \left(\cos\zeta(1-y)\sinh\zeta(1-y) - \sin\zeta(1-y)\cosh\zeta(1-y) \right) dy, \end{aligned}$$

and

$$\begin{aligned} & c_1 2\sin\zeta\sinh\zeta - c_3\left(\cos\zeta\sinh\zeta+\sin\zeta\cosh\zeta\right) \\ = \;& \frac{1}{4\zeta^3}\int_0^1 f\left(y\right)2\sin\zeta\left(1-y\right)\sinh\zeta\left(1-y\right)dy. \end{aligned}$$

Again, for symmetry reasons, we have $k_1\left(x,y,\lambda\right)=k_1\left(y,x,\lambda\right)$. So after solving the equations, the kernel is given by

$$\begin{aligned} k_1\left(x,y,\lambda\right) = \;& \frac{1}{4\zeta^3}\frac{1}{\sinh^2\zeta-\sin^2\zeta}\left(\left(\cos\zeta\sinh\zeta+\sin\zeta\cosh\zeta\right)\right. \\ & \times\left(\sin\zeta x\cosh\zeta x-\cos\zeta x\sinh\zeta x\right) \\ & \times\left(\sin\zeta\left(1-y\right)\cosh\zeta\left(1-y\right)-\cos\zeta\left(1-y\right)\sinh\zeta\left(1-y\right)\right) \\ & +\left(\sin\zeta\cosh\zeta-\cos\zeta\sinh\zeta\right)\,2\sin\zeta x\sinh\zeta x \\ & \times\sin\zeta\left(1-y\right)\sinh\zeta\left(1-y\right) \\ & +\sin\zeta\sinh\zeta\,\left(\sin\zeta x\cosh\zeta x-\cos\zeta x\sinh\zeta x\right) \\ & \times 2\sin\zeta\left(1-y\right)\sinh\zeta\left(1-y\right) \\ & \left.+2\sin\zeta x\sinh\zeta x\sin\zeta\left(1-y\right)\sinh\zeta\left(1-y\right)\right), \end{aligned} \tag{4}$$

for $0\le x\le y\le 1$.

From the equations of the kernels it is now possible to calculate the spectra, and in particular the first real eigenvalues $\lambda_0\left(A_i\right)$ $(i=1,2,3)$ of the operators. By solving the equation $\sinh^2\zeta=\sin^2\zeta$ for A_1 and the equation $\sinh^2\zeta=-\sin^2\zeta$ for A_2, we arrive at $\lambda_0\left(A_1\right)=\lambda_0$, $\lambda_0\left(A_2\right)=-\pi^4$, and $\lambda_0\left(A_3\right)=-\infty$.

4. Positivity of the kernels

Throughout this section we set $\zeta=4\lambda^{\frac{1}{4}}$.

4.1. **The physical operator.** To bring the kernel k_1 into a more manageable form we introduce new variables p and q via $p=\zeta\left(1+x-y\right)$ and $q=\zeta\left(1-x-y\right)$. The condition $0\le x\le y\le 1$ is then transformed to $|q|\le p\le\zeta$. The kernel can now be written as

$$k_1\left(x,y,\lambda\right)=\frac{1}{4\zeta^3}\frac{1}{\sinh^2\zeta-\sin^2\zeta}F_1\left(p,q,\zeta\right),$$

where

$$\begin{aligned} F_1\left(p,q,\zeta\right) = \;& 2\sin\zeta\sinh\zeta\left(\sin p\cosh q-\sin p\cosh q-\cos q\sinh p+\cos p\sinh p\right) \\ & +\cos\zeta\sinh\zeta\left(-\sin p\sinh p+\sin q\sinh q\right. \\ & \left.+\cos q\cosh q+\cos p\cosh p-2\cos p\cosh q\right) \\ & +\sin\zeta\cosh\zeta\left(-\sin p\sinh p+\sin q\sinh q\right. \\ & \left.-\cos q\cosh q-\cos p\cosh p-2\cos q\cosh p\right). \end{aligned} \tag{5}$$

Since $\sinh t\ge\sin t$ for all positive t and since $F_1\left(p,q,\zeta\right)=F_1\left(p,-q,\zeta\right)$, the kernel is positive if and only if $F_1\left(p,q,\zeta\right)\ge 0$ for $0\le q\le p\le\zeta$.

We now claim that the function $\left(q\mapsto F_1\left(p,q,\zeta\right)\right)$ is decreasing for $0\le q\le p\le\zeta\le\zeta_c$. In fact we calculate

$$\begin{aligned} \frac{\partial}{\partial q}F_1\left(p,q,\zeta\right) = \;& 2\sinh\zeta\sinh q\left(\cos\left(\zeta-q\right)-\cos\left(\zeta-p\right)\right) \\ & +\sin\zeta\sin q\left(\cosh\left(\zeta-q\right)-\cosh\left(\zeta-p\right)\right), \end{aligned} \tag{6}$$

which is negative due to Lemma 5.4. Together with the fact that $F_1(p,p,\zeta)=0$ this proves that $F_1(p,q,\zeta)$ is positive for $0\le q\le p\le\zeta\le\zeta_c$.

To show that F_1 is not positive if $\zeta>\zeta_c$ we define for a fixed ζ the function $\psi(p):=F_1(p,0,\zeta)$. Then

$$\begin{aligned}\psi'(p) &= 2\sin\zeta\sinh\zeta\,(2\cos p\cosh p-\cos p-\cosh p)\\ &\quad +2\cos\zeta\sinh\zeta\sin p\,(1-\cosh p)\\ &\quad +2\sin\zeta\cosh\zeta\sinh p\,(1-\cos p)\,,\end{aligned}\tag{7}$$

$$\begin{aligned}\psi''(p) &= 2\sin\zeta\sinh\zeta\,(2\cos p\sinh p-2\sin p\cosh p+\sin p-\sinh p)\\ &\quad +2\cos\zeta\sinh\zeta\,(\cos p-\cos p\cosh p-\sin p\sinh p)\\ &\quad +2\sin\zeta\cosh\zeta\,(\cosh p+\sin p\sinh p-\cos p\cosh p)\,,\end{aligned}\tag{8}$$

$$\begin{aligned}\psi'''(p) &= 2\sin\zeta\sinh\zeta\,(-2\sin p\sinh p+\cos p-\cosh p)\\ &\quad +2\cos\zeta\sinh\zeta\,(-\sin p-2\cos p\sinh p)\\ &\quad +2\sin\zeta\cosh\zeta\,(\sinh p+2\sin p\cosh p)\,,\end{aligned}\tag{9}$$

and

$$\begin{aligned}\psi''''(p) &= 2\sin\zeta\sinh\zeta\,(-2\sin p\cosh p-2\cos p\sinh p-\sin p-\sinh p)\\ &\quad +2\cos\zeta\sinh\zeta\,(-\cos p-2\cos p\cosh p+2\sin p\sinh p)\\ &\quad +2\sin\zeta\cosh\zeta\,(\cosh p+2\cos p\cosh p+2\sin p\sinh p)\,.\end{aligned}\tag{10}$$

From this we obtain immediately $\psi(0)=\psi'(0)=\psi''(0)=\psi'''(0)=0$ and

$$\psi''''(0)=6\,(\sin\zeta\cosh\zeta-\cos\zeta\sinh\zeta)\,,$$

which owing to Lemma 5.2 is negative for $\zeta_c<\zeta<\frac{3}{2}\pi$. So for small values of p and for $\zeta>\zeta_c$, $\psi(p)$ will be negative. This concludes the proof of the theorem for the case of the physical operator A_1.

4.2. **The square operator.** To improve the appearance of this kernel we will again use the transformation $p=\zeta(1+x-y)$ and $q=\zeta(1-x-y)$, where $|q|\le p\le\zeta$. The kernel then takes the form

$$k_2(x,y,\lambda)=\frac{1}{8\zeta^3}\frac{1}{\sin^2\zeta+\sinh^2\zeta}F_2(p,q,\zeta)\,,$$

with

$$\begin{aligned}F_2(p,q,\zeta) &= \cos\zeta\sinh\zeta\,(\cos p\cosh p-\sin p\sinh p\\ &\qquad -\cos q\cosh q+\sin q\sinh q)\\ &\quad +\sin\zeta\sinh\zeta\,(\cos p\cosh p+\sin p\sinh p\\ &\qquad -\cos q\cosh q-\sin q\sinh q)\,.\end{aligned}\tag{11}$$

As in the case of the physical operator, the kernel is positive if and only if $F_2(p,q,\zeta)$ is positive for $0\le q\le p\le\zeta$. The function F_2 can be written in the form

$$F_2(p,q,\zeta)=\psi(p,\zeta)-\psi(q,\zeta)\,,$$

where

$$\begin{aligned}\psi(p,\zeta) &= \cos\zeta\sinh\zeta\,(\cos p\cosh p-\sin p\sinh p)\\ &\quad +\sin\zeta\sinh\zeta\,(\cos p\cosh p+\sin p\sinh p)\,.\end{aligned}\tag{12}$$

From this we deduce that F_2 is positive if and only if the function $\psi(p,\zeta)$ is increasing in p for $0 \le p \le \zeta$. We calculate

$$\frac{\partial}{\partial p}\psi(p,\zeta) = 2(\sin\zeta\cosh\zeta\cos p\sinh p - \cos\zeta\sinh\zeta\sin p\cosh p),$$

which according to Lemma 5.3 is positive if $0 \le p \le \zeta \le \zeta_c$ and is negative for small p if $\zeta_c < \zeta < \frac{3}{2}\pi$.

4.3. **One-sided boundary conditions.** For the operator with one-sided boundary conditions it is immediate from the equation of the kernel k_3 that the kernel is positive if and only if the function

$$F_3(t) = \sin t\cosh t - \cos t\sinh t$$

is positive for $0 \le t \le \zeta$. According to Lemma 5.2 this is the case for $\zeta \le \zeta_c$ and this fails if $\zeta_c < \zeta < \frac{3}{2}\pi$.

5. Inequalities

In this section we prove the technical inequalities used in the previous section.

Lemma 5.1. *Let ζ_c be the first positive solution of $\tan z - \tanh z = 0$, and define the function g by $g(t) = \frac{\tan t}{tanht}$ for $t > 0$. Then the following hold:*
(i) $\pi < \zeta_c < \frac{3}{2}\pi$.
(ii) The function $g(t)$ is increasing everywhere where it is defined.
(iii) $\lim_{t\to 0} g(t) = 1$.
(iv) *For* $0 < s < t < \frac{\pi}{2}$ $\quad g(s) < g(t)$.
For $0 < s < \frac{\pi}{2} < t \le \zeta_c$ $\quad g(s) > g(t)$.
For $\frac{\pi}{2} < s < t < \zeta_c$ $\quad g(s) < g(t)$.
(v) For $\zeta_c < t < \frac{3}{2}\pi$ there exists a (small) $s > 0$ such that $g(s) < g(t)$.

Proof. Since $g'(t) = \frac{\sinh t\cosh t - \sin t\cos t}{\cos^2 t\cosh^2 t}$, and since $\sinh t > \sin t$ and $\cosh t > \cos t$ for all positive t g' is always positive for positive t, and (ii) follows.

Part (iii) is trivial.

Now for $t > 0$, $\tan t = \tanh t$ if and only if $g(t) = 1$. From (ii) and (iii) we deduce that $\frac{\pi}{2} < \zeta_c < \frac{3}{2}\pi$. And since $g(\pi) = 0$, we obtain that $\pi < \zeta_c$ and so (i) is proven.

Parts (iv) and (v) are immediate consequences of (ii) and (iii). □

Lemma 5.2. *The function f given by*

$$f(t) = \sin t\cosh t - \cos t\sinh t$$

is positive for $0 < t < \zeta_c$, and is negative for $\zeta_c < t \le 2\pi$.

Proof. We calculate $f(0) = 0$ and

$$f'(t) = 2\sin t\sinh t.$$

In particular f is increasing for $0 \le t \le \pi$ and decreasing for $\pi \le t \le 2\pi$. Therefore there is at most one solution t_0 of $f(t) = 0$ for $0 < t \le 2\pi$, and $f(t)$ must be positive for $0 < t < t_0$ and negative for $t_0 < t \le 2\pi$. But since $f(t) = \cos t\cosh t(\tan t - \tanh t)$, we see that $f(\zeta_c) = 0$ and therefore $t_0 = \zeta_c$. □

Lemma 5.3. *The function*

$$h\,(s,t) := \cos s \sinh s \sin t \cosh t - \sin s \cosh s \cos t \sinh t$$

is positive for $0 \le s \le t \le \zeta_c$. *For* $\zeta_c < t < \frac{3}{2}\pi$ *there exists a (small)* $s > 0$ *such that* $h\,(s,t) < 0$.

Proof. If $s = \frac{\pi}{2}$ or $t = \frac{\pi}{2}$, the claim is true. Now if $s \ne \frac{\pi}{2}$ and $t \ne \frac{\pi}{2}$, we write

$$\begin{aligned} h\,(s,t) &= \left(\frac{\tan t}{\tanh t} - \frac{\tan s}{\tanh s}\right) \cos s \cos t \sinh s \sinh t \\ &= (g\,(t) - g\,(s)) \cos s \cos t \sinh s \sinh t, \end{aligned}$$

where g is defined as in Lemma 5.1. We remark that $\sinh t$ is always positive for $t > 0$. Now using Lemma 5.1 we get

- If $0 < s < t < \frac{\pi}{2}$, then $g\,(t) - g\,(s)$ is positive, $\cos s$ and $\cos t$ are also positive, and so $h\,(s,t)$ is positive.
- If $0 < s < \frac{\pi}{2} < t \le \zeta_c$, then $g\,(t) - g\,(s)$ is negative, $\cos s$ is positive, and $\cos t$ is negative. Therefore $h\,(s,t)$ is positive.
- If $\frac{\pi}{2} < s < t < \zeta_c$, then $g\,(t) - g\,(s)$ is positive, $\cos s$ and $\cos t$ are negative, and again $h\,(s,t)$ is positive.
- If $\zeta_c < t < \frac{3}{2}\pi$, then there is a small $s > 0$ such that $g\,(t) - g\,(s)$ is positive, $\cos s$ is positive, and $\cos t$ is negative. So for small s, $h\,(s,t)$ is negative.

□

Lemma 5.4. *For fixed* s *and* t *satisfying* $0 \le s \le t \le \zeta_c$ *the function*

$$k\,(x) := \sinh s \sinh t \cos x + \sin s \sin t \cosh x$$

is a decreasing function in x *for* $0 \le x \le t - s$.

Proof. The derivative of k is given by

$$k'\,(x) = \sin s \sin t \sinh x - \sinh s \sinh t \sin x.$$

To prove the claim it is necessary to show that $k'\,(x) \le 0$ for $0 \le x \le t - s$. This is equivalent with

$$\frac{\sin s}{\sinh s} \frac{\sin t}{\sinh t} \le \frac{\sin x}{\sinh x}.$$

Since $\frac{d}{dt}\frac{\sin t}{\sinh t} = \frac{\cos t \sinh t - \sin t \cosh t}{\sinh^2 t}$, which is negative by Lemma 5.2, it is sufficient to show that

$$\frac{\sin s}{\sinh s} \frac{\sin t}{\sinh t} \le \frac{\sin\,(t-s)}{\sinh\,(t-s)},$$

or if the right-hand side is expanded and the divisors are multiplied,

$$\sin t \sinh t\,(\sin s \cosh s - \cos s \sinh s) \le \sin s \sinh s\,(\sin t \cosh t - \cos t \sinh t)\,.$$

By Lemma 5.2 we see that $\sin t \cosh t - \cos t \sinh t$ is positive and we obtain the condition

$$\frac{\sin t \sinh t}{\sin t \cosh t - \cos t \sinh t} \le \frac{\sin s \sinh s}{\sin s \cosh s - \cos s \sinh s}.$$

So it is sufficient to prove that the function $\left(t \mapsto \frac{\sin t \sinh t}{\sin t \cosh t - \cos t \sinh t}\right)$ is decreasing. An easy calculation shows

$$\frac{d}{dt}\left(\frac{\sin t \sinh t}{\sin t \cosh t - \cos t \sinh t}\right) = \frac{\sin^2 t - \sinh^2 t}{(\sin t \cosh t - \cos t \sinh t)^2},$$

which is negative. This concludes the proof. □

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