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A UNIFORM REFINEMENT PROPERTY FOR CONGRUENCE LATTICES

FRIEDRICH WEHRUNG

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Abstract. The Congruence Lattice Problem asks whether every algebraic distributive lattice is isomorphic to the congruence lattice of a lattice. It was hoped that a positive solution would follow from E. T. Schmidt's construction or from the approach of P. Pudlák, M. Tischendorf, and J. Tůma. In a previous paper, we constructed a distributive algebraic lattice A with $\aleph_2$ compact elements that cannot be obtained by Schmidt's construction. In this paper, we show that the same lattice A cannot be obtained using the Pudlák, Tischendorf, Tůma approach.

The basic idea is that every congruence lattice arising from either method satisfies the *Uniform Refinement Property*, that is not satisfied by our example. This yields, in turn, corresponding negative results about congruence lattices of sectionally complemented lattices and two-sided ideals of von Neumann regular rings.

Introduction

E. T. Schmidt introduces in [11] the notion of a *weakly distributive* (resp., *distributive*) homomorphism of semilattices, and proves the following important result (see, for example, [11, Satz 8.1] and [12, Theorem 3.6.9]):

Schmidt's Lemma. *Let A be an algebraic distributive lattice. If the semilattice of compact elements of A is the image under a distributive homomorphism of a generalized Boolean semilattice, then there exists a lattice L such that* $\operatorname{Con} L$ *is isomorphic to A.*

This result yields important partial positive answers to the Congruence Lattice Problem (see, for example, [9, 14] for a survey). On the other hand, we prove [15, Theorem 2.15], which implies the following result:

Theorem. *For any cardinal number $\kappa \geq \aleph_2$, there exists a distributive semilattice S_κ of size κ that is not a weakly distributive image of any distributive lattice.*

In this paper, we introduce a monoid-theoretical property, the *Uniform Refinement Property*, that *is not* satisfied by the semilattices S_κ of the theorem above,

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but it *is* satisfied by the congruence semilattice of any lattice satisfying an additional condition, the *Congruence Splitting Property* (Theorem 3.3). The latter is satisfied by any lattice which is either sectionally complemented, or relatively complemented, or a direct limit of atomistic lattices (Proposition 3.2). In particular, for all $\kappa \geq \aleph_2$, the semilattice S_κ of the above theorem is not isomorphic to the congruence semilattice of any sectionally complemented lattice; this gives a partial negative answer to [8, Problem II.8]. It follows that no semilattice S_κ can be realized as the semilattice of finitely generated two-sided ideals of a von Neumann regular ring.

Notation and terminology. The *refinement property* is the monoid-theoretical axiom stating that for every equation of the form $a_0 + a_1 = b_0 + b_1$, there exist elements c_{ij} $(i, j < 2)$ such that for all $i < 2$, $a_i = c_{i0} + c_{i1}$ and $b_i = c_{0i} + c_{1i}$. All semilattices will be join-semilattices (not necessarily bounded). A semilattice is, as usual, *distributive*, if it satisfies the refinement property.

If u and v are elements of a lattice L, then $\Theta_L(u, v)$ (or $\Theta(u, v)$ if there is no ambiguity) denotes the least congruence of L identifying u and v. Furthermore, $\operatorname{Con} L$ (resp., $\operatorname{Con}_{\mathrm{c}} L$) denotes the lattice (resp., semilattice) of all congruences (resp., compact congruences) of L. If L has a least element (always denoted by 0), an *atom* of L is a minimal element of $L \setminus \{0\}$. A lattice L with zero is *atomistic*, if every element of L is a *finite* join of atoms.

Our lattice-theoretical results will be applied to rings in Section 4. All our rings are *associative* and *unital* (but not necessarily commutative). Recall that a ring R is (von Neumann) *regular*, if it satisfies the axiom $(\forall x)(\exists y)(xyx = x)$. If R is a regular ring, then every principal right ideal of R is of the form eR, where e is idempotent, and the set $\mathcal{L}(R)$ of all principal right ideals of R, partially ordered by inclusion, is a complemented modular lattice.

1. Weak-distributive homomorphisms

We shall first modify slightly the original definition, due to E. T. Schmidt [12], [13] of a weakly distributive homomorphism. A homomorphism of semilattices $f\colon S \to T$ is *weakly distributive* at an element u of S, if for all $y_0, y_1 \in T$ such that $f(u) = y_0 + y_1$, there are $x_0, x_1 \in S$ such that $x_0 + x_1 = u$ and $f(x_i) \leq y_i$, for all $i < 2$. Say that f is *weakly distributive*, if it is weakly distributive at every element of S.

In particular, if f is surjective, one recovers the usual definition of a weakly distributive homomorphism. Furthermore, note that with this new definition, any composition of two weakly distributive homomorphisms remains weakly distributive.

Lemma 1.1. *Let $f\colon S \to T$ be a homomorphism of semilattices, with T distributive. Then the set of all elements of S at which f is weakly distributive is closed under join.*

Proof. Let X be the set of all elements of S at which f is weakly distributive. It suffices to prove that if u' and u'' are any two elements of X, then $u = u' + u''$ belongs to X. Thus let y_i $(i < 2)$ be elements of T such that $f(u) = y_0 + y_1$, that is, since f is a homomorphism of semilattices, $f(u') + f(u'') = y_0 + y_1$ holds. Since T is distributive, it satisfies the refinement property; thus there are decompositions $y_i = y_i' + y_i''$ $(i < 2)$ such that $f(u') = y_0' + y_1'$ and $f(u'') = y_0'' + y_1''$. Since both u'

and u'' belong to X, there are decompositions $u' = x_0' + x_1'$ and $u'' = x_0'' + x_1''$ such that $f(x_i') \leq y_i'$ and $f(x_i'') \leq y_i''$ $(i < 2)$. Put $x_i = x_i' + x_i''$ $(i < 2)$. Then $f(x_i) \leq y_i$ and $x_0 + x_1 = u$. Therefore, $u \in X$. □

The following result yields a large class of weakly distributive homomorphisms:

Proposition 1.2. *Let* $e\colon K \to L$ *be a lattice homomorphism with convex range. Then the induced semilattice homomorphism* $f\colon \operatorname{Con_c} K \to \operatorname{Con_c} L$ *is weakly distributive.*

Proof. It is well-known that $\operatorname{Con_c} L$ is a distributive semilattice (see, for example, [8, Theorem II.3.11]). Hence, by Lemma 1.1, it suffices to prove that f is weakly distributive at every α of the form $\Theta_K(u, v)$ where $u \leq v$ in K. Thus let β_i $(i < 2)$ be such that $f(\Theta_K(u, v)) = \beta_0 \vee \beta_1$, that is, $\Theta_L(f(u), f(v)) = \beta_0 \vee \beta_1$. Thus, by [8, Lemma III.1.3], there exist a positive integer n and elements w_i' $(i \leq 2n)$ of L such that $f(u) = w_0' \leq w_1' \leq \cdots \leq w_{2n}' = f(v)$ and for all $i < n$, $w_{2i}' \equiv w_{2i+1}' \pmod{\beta_0}$ and $w_{2i+1}' \equiv w_{2i+2}' \pmod{\beta_1}$. But the range of e is convex in L; thus the w_i' belong to the range of e; thus there are elements $w_i \in K$ such that $e(w_i) = w_i'$. One can of course take $w_0 = u$ and $w_{2n} = v$, and, after replacing each w_i by $\bigvee_{j \leq i}(w_j \vee u) \wedge v$, one can suppose that $u = w_0 \leq w_1 \leq \cdots \leq w_{2n} = v$. Then put $\alpha_0 = \bigvee_{i<n} \Theta_K(w_{2i}, w_{2i+1})$ and $\alpha_1 = \bigvee_{i<n} \Theta_K(w_{2i+1}, w_{2i+2})$. We have $\alpha_0 \vee \alpha_1 = \Theta_K(u, v) = \alpha$, and $f(\alpha_0) = \bigvee_{i<n} \Theta_L(f(w_{2i}), f(w_{2i+1})) \subseteq \beta_0$; similarly, $f(\alpha_1) \subseteq \beta_1$. □

2. The Uniform Refinement Property

In order to illustrate the terminology of the section title, let us first consider any equation system (in a given semilattice) of the form

$$\Sigma\colon a_i + b_i = \text{constant} \qquad (\text{for all } i \in I).$$

When $I = \{i, j\}$, a satisfactory notion of a *refinement* of Σ consists of four elements c_{ij}^{uv} $(u, v < 2)$ satisfying the equations

$$\begin{aligned} a_i &= c_{ij}^{00} + c_{ij}^{01} &\quad\text{and}\quad&& b_i &= c_{ij}^{10} + c_{ij}^{11}, \\ a_j &= c_{ij}^{00} + c_{ij}^{10} &\quad\text{and}\quad&& b_j &= c_{ij}^{01} + c_{ij}^{11}, \end{aligned} \tag{2.1}$$

see Figure 1.

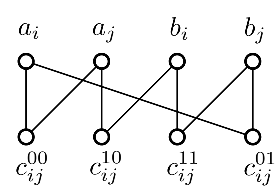

Figure 1

Note that (2.1) implies immediately the following consequence:

$$a_i \leq a_j + c_{ij}^{01}. \tag{2.2}$$

When I is an arbitrary finite set, one can extend this in a natural way and thus define a refinement of Σ to be a $\mathcal{P}(I)$-indexed family of elements of S satisfying suitable generalizations of (2.1). Nevertheless, this cannot be extended immediately to the infinite case, so that we shall focus instead on the consequence (2.2) of refinement, together with an additional "coherence condition" $c_{ik}^{01} \leq c_{ij}^{01} + c_{jk}^{01}$. More precisely, we state the following definition:

Definition 2.1. Let S be a semilattice, and let e be an element of S. Say that the *Uniform Refinement Property* holds at e, if for all families $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ of elements of S such that $(\forall i \in I)(a_i + b_i = e)$, there are families $(a_i^*)_{i\in I}$, $(b_i^*)_{i\in I}$ and $(c_{ij})_{(i,j)\in I\times I}$ of elements of S satisfying the following properties:

(i) For all $i \in I$, $a_i^* \leq a_i$ and $b_i^* \leq b_i$ and $a_i^* + b_i^* = e$.
(ii) For all $i,\ j \in I$, $c_{ij} \leq a_i^*, b_j^*$ and $a_i^* \leq a_j^* + c_{ij}$.
(iii) For all $i,\ j,\ k \in I$, $c_{ik} \leq c_{ij} + c_{jk}$.

Say that S satisfies the Uniform Refinement Property, if the Uniform Refinement Property holds at every element of S.

It is to be noted that this is far from being the only possible "reasonable" definition for a "Uniform Refinement Property"; see [15, Theorem 2.8, Claim 1], which suggests a quite different Uniform Refinement Property (in the context of partially ordered vector spaces). The following result allows us to focus the investigation on a generating set in the case where our semilattice is distributive:

Proposition 2.2. *Let S be a distributive semilattice. Then the set X of all elements of S at which the Uniform Refinement Property holds is closed under join.*

Proof. Let e_0 and e_1 be two elements of X, and put $e = e_0 + e_1$. Let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be two families of elements of S such that for all $i \in I$, $a_i + b_i = e$. Since S is distributive, there are decompositions $a_i = a_{i0} + a_{i1}$, $b_i = b_{i0} + b_{i1}$ (for all $i \in I$) such that $e_\nu = a_{i\nu} + b_{i\nu}$ (for all $i \in I$ and $\nu < 2$). Since both e_0 and e_1 belong to X, to the latter decompositions correspond elements $a_{i\nu}^*$, $b_{i\nu}^*$ and $c_{ij\nu}$ ($i,\ j \in I$ and $\nu < 2$) witnessing the Uniform Refinement Property at e_0 and e_1.

Now put $a_i^* = a_{i0}^* + a_{i1}^*$, $b_i^* = b_{i0}^* + b_{i1}^*$ and $c_{ij} = c_{ij0} + c_{ij1}$. It is obvious that (i) to (iii) of Definition 2.1 above are satisfied with respect to $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$, thus proving that $X + X \subseteq X$. □

Note also the following result, whose easy proof we shall omit:

Proposition 2.3. *Let $f\colon S \to T$ be a weakly distributive homomorphism of semilattices and let $u \in S$. If the Uniform Refinement Property holds at u in S, then it also holds at $f(u)$ in T.* □

3. Congruence splitting lattices; property (C)

We will say throughout this paper that a lattice L is *congruence splitting*, if for all $a \leq b$ in L and all congruences α_0 and α_1 of L such that $\Theta(a, b) \subseteq \alpha_0 \vee \alpha_1$, there are elements x_0 and x_1 of $[a, b]$ such that $x_0 \vee x_1 = b$ and for all $i < 2$, $\Theta(a, x_i) \subseteq \alpha_i$. Of course, it suffices to consider the case where both α_0 and α_1 are compact congruences and $\Theta(a, b) = \alpha_0 \vee \alpha_1$.

For all elements a, b and c of a given lattice, write $a \lessdot_c b$, if there exists x such that $a \vee x = b$ and $a \wedge x \leq c$.

Definition 3.1. A lattice L has *property (C)*, if for all $a \leq b$ and all c in L, there exist a positive integer n and elements x_i ($i \leq n$) of L such that $a = x_0 \lessdot_c x_1 \lessdot_c \cdots \lessdot_c x_n = b$.

The letter ‘C’ stands here for ‘complement’. In the following proposition, we record a few elementary properties of lattices either with property (C) or congruence splitting.

Proposition 3.2. *The following properties hold:*

(a) *Every relatively complemented lattice, or every sectionally complemented lattice, has property (C).*
(b) *Every atomistic lattice has property (C).*
(c) *The class of all lattices satisfying property (C) is closed under direct limits.*
(d) *Every lattice satisfying property (C) is congruence splitting.*
(e) *The class of all congruence splitting lattices is closed under direct limits.*

Proof. (a) is obvious (take $n = 1$).

(b) Let L be an atomistic lattice. Note then that for every $a \in L$ and every atom p, $a \lessdot_0 a \vee p$: indeed, if $p \leq a$, then in fact $a = a \vee p$, and otherwise, $a \wedge p = 0$. Now let $a < b$ in L. There exist a positive integer n and atoms p_i $(i \leq n)$ such that $b = \bigvee_{i \leq n} p_i$. For all $i \leq n$, put $c_i = a \vee \bigvee_{j \leq i} p_j$. Then, by previous remark, $a = c_0 \lessdot_0 c_1 \lessdot_0 \cdots \lessdot_0 c_n = b$.

(c) is straightforward.

(d) Let L be a lattice satisfying property (C), let α_j $(j < 2)$ be congruences of L, and let $a \leq b$ in L such that $\Theta(a, b) \subseteq \alpha_0 \vee \alpha_1$. To reach the desired conclusion, we argue by induction on the minimal length $n = n(a, b)$ of a chain $a = c_0 \lessdot_a c_1 \lessdot_a \cdots \lessdot_a c_n = b$ such that for all $i < n$, there exists $j < 2$ such that $c_i \equiv c_{i+1} \pmod{\alpha_j}$ (such a chain exists by [8, Lemma III.1.3] and property (C)). If $a < b$, then there exists c such that $a \leq c$, $n(a, c) < n(a, b)$, $c \lessdot_a b$, and, without loss of generality, $c \equiv b \pmod{\alpha_0}$. By induction hypothesis, there are $y_j \in [a, c]$ $(j < 2)$ such that $y_0 \vee y_1 = c$ and $y_j \equiv a \pmod{\alpha_j}$. Furthermore, there exists z such that $z \wedge c \leq a$ and $z \vee c = b$. Put $x_0 = y_0 \vee z$ and $x_1 = y_1$; then $x_j \in [a, b]$, $x_j \equiv a \pmod{\alpha_j}$, and $x_0 \vee x_1 = b$.

(e) Let L be a direct limit of a direct system $(L_i, e_{ij})_{i \leq j \text{ in } I}$ of lattices and lattice homomorphisms (I is a directed poset), with limiting maps $e_i \colon L_i \to L$ (for all $i \in I$). It is well-known that $\operatorname{Con_c} L$ is then the direct limit of the $\operatorname{Con_c} L_i$ with the corresponding transition maps and limiting maps. Once this observation is made, it is routine to verify that if all the L_i are congruence splitting, then so is L. □

Theorem 3.3. *Let L be a congruence splitting lattice. Then* $\operatorname{Con_c} L$ *satisfies the Uniform Refinement Property.*

Proof. Put $S = \operatorname{Con_c} L$. By Proposition 2.2, it suffices to prove that S satisfies the Uniform Refinement Property at every element of S of the form $\varepsilon = \Theta(u, v)$ where $u \leq v$ in L. Thus let $(\alpha_i)_{i \in I}$ and $(\beta_i)_{i \in I}$ be two families of elements of S such that for all $i \in I$, $\alpha_i + \beta_i = \varepsilon$. Since L is congruence splitting, there are elements s_i and t_i of $[u, v]$ such that $s_i \vee t_i = v$ and $\Theta(u, s_i) \subseteq \alpha_i$ and $\Theta(u, t_i) \subseteq \beta_i$. Now, for all $i, j \in I$, put

$$\alpha_i^* = \Theta(u, s_i), \qquad \beta_i^* = \Theta(u, t_i) \qquad \text{and} \qquad \gamma_{ij} = \Theta(s_j, s_i \vee s_j).$$

It is immediate that $\alpha_i^* \subseteq \alpha_i$, $\beta_i^* \subseteq \beta_i$ and $\alpha_i^* + \beta_i^* = \varepsilon$. Furthermore, for all $i, j \in I$, we have $\gamma_{ij} \subseteq \Theta(u, s_i) = \alpha_i^*$ and $\gamma_{ij} \subseteq \Theta(u, t_j) = \beta_j^*$. Finally, γ_{ij} is the least congruence θ of L such that $\theta(s_i) \leq \theta(s_j)$; it follows immediately that $\gamma_{ik} \leq \gamma_{ij} + \gamma_{jk}$, thus completing the proof. □

On the other hand, an inspection of the proof of the theorem stated in the Introduction [15, Theorem 2.15] shows in fact the following property of the corresponding semilattices S_κ:

Lemma 3.4. *For every cardinal number $\kappa \geq \aleph_2$, the semilattice S_κ does not satisfy the Uniform Refinement Property at its largest element.* $\square$

By putting together Proposition 2.3 and Lemma 3.4, one deduces the following:

Corollary 3.5. *For every cardinal number $\kappa \geq \aleph_2$, there are no congruence splitting lattice L and no weakly distributive semilattice homomorphism $\mu\colon \operatorname{Con_c} L \to S_\kappa$ with range containing the largest element of S_κ.* $\square$

Corollary 3.6. *Let $\kappa \geq \aleph_2$ be a cardinal number. Consider both following statements:*

(i) *There exist a lattice L and a weakly distributive homomorphism $\mu\colon \operatorname{Con_c} L \to S_\kappa$ with range containing (as an element) the largest element of S_κ.*

(ii) *For every bounded lattice L, there exists a congruence splitting lattice L' such that $\operatorname{Con_c} L \cong \operatorname{Con_c} L'$.*

Then (i) and (ii) cannot be simultaneously true.

Proof. Suppose that both (i) and (ii) are simultaneously true, and let L, μ be as in (i). Since L is the direct union of its closed intervals, we obtain, by using Proposition 1.2, the existence of a closed interval K of L such that the restriction of μ to K satisfies (i). Then, applying (ii) to K contradicts Corollary 3.5. $\square$

In particular, the Congruence Lattice Problem and the problem whether every lattice has a congruence-preserving embedding into a sectionally complemented lattice cannot both have positive answers.

4. Applications to von Neumann regular rings

For any ring R, we denote by $\mathrm{FP}(R)$ the class of all *finitely generated projective* right R-modules, and by $\mathbf{V}(R)$ the (commutative) monoid of isomorphism classes of elements of $\mathrm{FP}(R)$, the addition of $\mathbf{V}(R)$ being defined by $[A] + [B] = [A \oplus B]$ ($[A]$ denotes the isomorphism class of A). Then $\mathbf{V}(R)$ is *conical*, that is, it satisfies the axiom $(\forall x, y)(x + y = 0 \Rightarrow x = y = 0)$. Moreover, if R is regular, then $\mathbf{V}(R)$ satisfies the refinement property. Furthermore, we equip $\mathbf{V}(R)$ with its *algebraic* preordering $\leq$, defined by $x \leq y$ if and only if there exists z such that $x + z = y$. References about this can be found in [1], [5], [6], [10].

Lemma 4.1. *Let R be a ring, let $J \in \operatorname{Id} R$, let a and b be idempotent elements of R such that $aR \cong bR$ (as right R-modules). Then $a \in J$ if and only if $b \in J$.*

Proof. Since $aR \cong bR$ and a and b are idempotent, there are elements $x \in aRb$ and $y \in bRa$ such that $a = xy$ and $b = yx$. Suppose, for example, that $a \in J$. Then $x \in aRb \subseteq J$ (because J is a right ideal of R); thus $b = yx \in J$ (because J is a left ideal of R). $\square$

If L is any lattice with 0, an ideal $\mathfrak{a}$ of L is *neutral*, if $x \in \mathfrak{a}$ and $x \sim y$ implies that $y \in \mathfrak{a}$ ($\sim$ is the relation of perspectivity). We will denote by $\operatorname{NId} L$ the lattice of all neutral ideals of L. Recall [3], [8] that if L is a sectionally complemented modular lattice, then $\operatorname{Con} L$ and $\operatorname{NId} L$ are (canonically) isomorphic.

Lemma 4.2. *Let R be a regular ring. Then an ideal $\mathfrak{a}$ of $\mathcal{L}(R)$ is neutral if and only if it is closed under isomorphism (that is, $J \in \mathfrak{a}$ and $I \cong J$ implies $I \in \mathfrak{a}$).*

Proof. It is trivial that if $\mathfrak{a}$ is closed under isomorphism, then it is neutral. Conversely, suppose that $\mathfrak{a}$ is neutral. Let $I, J \in \mathcal{L}(R)$ such that $I \cong J$ and $J \in \mathfrak{a}$. There exists $I' \in \mathcal{L}(R)$ such that $(I \cap J) \oplus I' = I$. Since $I \cap J \leq J$ and $J \in \mathfrak{a}$, we have $I \cap J \in \mathfrak{a}$. Furthermore, there exists $J' \leq J$ such that $I' \cong J'$. Since $I' \cap J = \{0\}$, we also have $I' \cap J' = \{0\}$; thus, by [5, Proposition 4.22], $I' \sim J'$. Since $J' \leq J$ and $J \in \mathfrak{a}$, we have $J' \in \mathfrak{a}$; thus, since $\mathfrak{a}$ is neutral, $I' \in \mathfrak{a}$. Hence, $I = (I \cap J) \oplus I' \in \mathfrak{a}$. $\square$

We denote by $\operatorname{Id} R$ the (algebraic) lattice of two-sided ideals of any ring R, and by $\operatorname{Id_c} R$ the semilattice of all compact (that is, finitely generated) elements of $\operatorname{Id} R$.

Theorem 4.3. *Let R be a regular ring. Then one can define two mutually inverse isomorphisms by the following rules:*

$$\varphi\colon \operatorname{NId}\mathcal{L}(R) \to \operatorname{Id} R, \qquad \mathfrak{a} \mapsto \{x \in R\colon xR \in \mathfrak{a}\},$$

and

$$\psi\colon \operatorname{Id} R \to \operatorname{NId}\mathcal{L}(R), \qquad I \mapsto \{J \in \mathcal{L}(R)\colon J \subseteq I\}.$$

Proof. First of all, we must verify that for all $\mathfrak{a} \in \operatorname{NId}\mathcal{L}(R)$, $\varphi(\mathfrak{a})$ as defined above is a two-sided ideal of R. It is obvious that $\varphi(\mathfrak{a})$ is an additive subgroup of R. Let $x \in \varphi(\mathfrak{a})$ and $\lambda \in R$. Then $[x\lambda R] \leq [xR]$ and, since R is regular, $[\lambda xR] \leq [xR]$ (the natural surjective homomorphism $xR \to \lambda xR$ splits). Therefore, it is sufficient to prove that if $J \in \mathfrak{a}$ and $[I] \leq [J]$, then $I \in \mathfrak{a}$. However, this results immediately from Lemma 4.2. Conversely, the fact that ψ takes its values in $\operatorname{NId}\mathcal{L}(R)$ results immediately from Lemma 4.1. The verification of the fact that both φ and ψ are order-preserving and mutually inverse is straightforward. $\square$

Corollary 4.4. *For any regular ring R,* $\operatorname{Con_c}\mathcal{L}(R)$ *is isomorphic to* $\operatorname{Id_c} R$. $\square$

In [2], G. M. Bergman proves among other things that any *countable* bounded distributive semilattice is isomorphic to $\operatorname{Id_c} R$ for some regular (and even ultramatricial) ring R. By Corollaries 3.5 and 4.4, one cannot generalize this to semilattices of size $\aleph_2$ (the size $\aleph_1$ case is still open):

Corollary 4.5. *For any cardinal number $\kappa \geq \aleph_2$, S_κ is not isomorphic to the semilattice of finitely generated two-sided ideals of any regular ring.* $\square$

Note that this also provides a negative answer to [15, Problem 2.16], *via* the following easy result:

Proposition 4.6. *Let R be a regular ring. Then* $\operatorname{Id_c} R$ *is isomorphic to the maximal semilattice quotient of* $\mathbf{V}(R)$.

Proof. Let $\pi : \mathbf{V}(R) \to \mathcal{P}(R)$ be the mapping defined by the rule

$$\pi(\alpha) = \{x \in R\colon [xR] \leq n\alpha \text{ for some positive integer } n\}.$$

Using the refinement property of $\mathbf{V}(R)$, it is easy to verify that π is a monoid homomorphism taking its values in $\operatorname{Id} R$. Moreover, for every $I \in \mathcal{L}(R)$, we have $\pi([I]) = RI$ and those elements generate $\operatorname{Id_c} R$; thus π maps $\mathbf{V}(R)$ onto $\operatorname{Id_c} R$. Finally, by [5, Corollary 2.23], for all $I,\ J \in \mathcal{L}(R)$, $\pi([I]) \leq \pi([J])$ if and only if there exists a positive integer n such that $[I] \leq n[J]$; again by using refinement, it follows that for all elements $\alpha,\ \beta \in \mathbf{V}(R)$, $\pi(\alpha) \leq \pi(\beta)$ if and only if there exists a positive integer n such that $\alpha \leq n\beta$. The conclusion follows. $\square$

Note added. Recently, the author, in a joint paper with M. Ploščica and J. Tůma, has proved that there exists a bounded lattice L such that $\operatorname{Con} L$ is not isomorphic to $\operatorname{Con} L'$, for any congruence splitting lattice L'. Compare this with Corollary 3.6 of this paper.

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Département de Mathématiques, Université de Caen, 14032 Caen Cedex, France
E-mail address: gremlin@math.unicaen.fr