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THE RATIONAL MAPS $z \mapsto 1 + 1/\omega z^d$ HAVE NO HERMAN RINGS

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ABSTRACT. We prove that for every $d \in \mathbb{N}, d \geq 2$, the rational maps in the family $\{z \mapsto 1 + 1/\omega z^d : \omega \in \mathbf{C} \setminus \{0\}\}$ have no Herman rings. From this we conclude a dynamical characterization for the parameters in the Mandelbrot set of these families. Further, we show that hyperbolic maps are dense in this family if and only if the set of parameters for which the Julia set is the whole sphere has no interior.

1. INTRODUCTION

It is well known that the dynamics of a rational map $R(z) = P(z)/Q(z)$ is very much influenced by the forward orbits of its critical points. Natural families of rational maps are defined by imposing conditions on these forward orbits. For example the polynomial maps are the rational maps that have a fixed critical point of maximal degree.

For quadratic rational maps, J.Milnor [M] suggested considering, among others, the family of those maps that take one critical point to the other. In appropriate coordinates these maps take the form $z \mapsto 1 + 1/\omega z^2$.

Before the work of Milnor, the family $\mathcal{F}_2 = \{z \mapsto 1 + 1/\omega z^2 : \omega \in \mathbf{C} \setminus \{0\}\}$ was considered by M.Lyubich [L]. He asked whether the maps in this family have Herman rings. M.Shishikura [Sh], using quasi-conformal surgery techniques, proved that the quadratic rational maps have no Herman rings, thus solving Lyubich's question as a particular case.

In this work we consider the families $\mathcal{F}_d = \{z \mapsto 1 + 1/\omega z^d : \omega \in \mathbf{C} \setminus \{0\}\}$, with $d \in \mathbb{N}, d \geq 2$. The family $\mathcal{F}_d$ is a normal form for the set of rational maps of degree d which have exactly two critical points, one of which maps onto the other under one iteration.

Although Shishikura's analytical methods might also imply that the rational maps in these families have no Herman rings, we have established this result through purely topological methods. Our main result is:

Theorem 1. *The rational maps in the family $\mathcal{F}_d$ have no Herman rings.*

The Mandelbrot set of a family of rational maps $\{f_\omega\}$, $\mathcal{M}(\{f_\omega\})$, is defined as the closure of the set $\{\omega : J(f_\omega)\text{is connected}\}$. Here $J(f_\omega)$ denotes the Julia set of f_ω.

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In [Y], Y.Yongcheng uses Shishikura's result to characterize dynamically the set of quadratic rational maps with connected Julia set.

From Theorem 1 we derive the following characterization theorem:

Theorem 2. *The set $\mathcal{M}(\mathcal{F}_d)$ is equal to the set $\{\ \omega : f_\omega$ has no attracting fixed point$\}$. Further, $\mathcal{M}(\mathcal{F}_d)$ is connected; its boundary is a regular analytic curve except for one singular point, namely the unique parameter in $\mathcal{M}(\mathcal{F}_d)$ where the Julia set is disconnected.*

In this theorem, and in what follows, $f_\omega(z)$ denotes $1 + 1/\omega z^d$.

We prove Theorem 1 in Section 2, while Theorem 2 is proved in Section 3. In Section 3 also we point out the fact that Theorem 1 allows us to reduce the

HD-conjecture. $\{\omega : f_\omega$ is hyperbolic $\}$ is dense in $\mathbf{C}$,

to the following one:

Conjecture. $\{\omega : J(f_\omega) = \overline{\mathbf{C}}\}$ has empty interior.

2. Proof of Theorem 1

Assume that for some $\omega \in \mathbf{C} \setminus \{0\},\ f_\omega(z) = 1 + 1/\omega z^d$ has a periodic cycle of Herman rings $\{H_0, H_1, \cdots, H_{m-1}\}$. Let $\mathcal{C}_0 \subset H_0$ be an f_ω^m-invariant analytic Jordan curve and define $\mathcal{C}_j = f_\omega^j(\mathcal{C}_0),\ j = 1, \cdots, m-1$. Let ν be the $d-$th root of unity $e^{2\pi i/d}$.

For a Jordan curve $\mathcal{C}$ in $\mathbf{C}$ denote by $I(\mathcal{C})$ the bounded component of $\mathbf{C} \setminus \mathcal{C}$ and by $E(\mathcal{C})$ the unbounded component. Call them respectively the **interior** and the **exterior** of $\mathcal{C}$.

No $\mathcal{C}_j$ passes through the critical values 1 and ∞ because the boundary of a Herman ring is contained in the closure of the forward orbit of the critical set $\{0, \infty\}$. Since each $\mathcal{C}_j$ is mapped injectively onto $\mathcal{C}_{j+1}$ and both critical points are of maximal degree, we have that for every $j = 0, \cdots, m-1,\ f_\omega^{-1}(\mathcal{C}_{j+1}) \pmod{m}$ is the disjoint union of the sets $\nu^k \mathcal{C}_j, k = 0, 1, \cdots, d-1$.

We have that $0 \in E(\mathcal{C}_j)\ \forall j = 1, \cdots, m-1$. In fact, if $0 \in I(\mathcal{C}_j)$ for some j, then $\nu\mathcal{C}_j \cap \mathcal{C}_j \neq \emptyset$ which contradicts the disjointness of the union $\bigcup_{k=0}^{k=d-1} \nu^k \mathcal{C}_j$.

It then follows that for every $j,\ 0 \in E(\nu^k \mathcal{C}_j)\ \forall k = 0, \cdots, d-1$. Also $\nu^k \mathcal{C}_j \subseteq E(\nu^l \mathcal{C}_j)$ for $k \neq l$. Since 0 is mapped to ∞ with multiplicity d, we conclude that

$$f_\omega : \bigcap_{k=0}^{k=d-1} E(\nu^k \mathcal{C}_j) \cup \{\infty\} \longrightarrow E(\mathcal{C}_{j+1}) \cup \{\infty\}$$

is a d-ramified covering. Hence, f_ω maps bijectively each $I(\nu^k \mathcal{C}_j),\ k = 0, 1, \cdots, d-1$, onto $I(\mathcal{C}_{j+1})$.

In particular, for every $j = 0, \cdots, m-1,\ f_\omega(I(\mathcal{C}_j)) = I(\mathcal{C}_{j+1}) \pmod{m}$. Therefore:

$$\bigcup_{n \in \mathbb{N}} f_\omega^n(I(\mathcal{C}_0)) = \bigcup_{j=0}^{j=m-1} I(\mathcal{C}_j).$$

Then, by Montel's Theorem, $I(\mathcal{C}_0)$ belongs to the Fatou set of f_ω. But this is impossible because $\mathcal{C}_0$ is an f_ω^m-invariant Jordan curve of a cycle of Herman rings.

3. Proof of Theorem 2

For rational maps, the Julia set is connected if and only if every component of the Fatou set is simply connected. We would like then to obtain a dynamical characterization for the parameters in the set $\{\ \omega : \text{every component of } F(f_\omega) \text{ is simply connected}\ \}$.

From Theorem 1, and the classification of the periodic Fatou components, it follows that every periodic component of $F(f_\omega)$ is either attractive, parabolic or Siegel disk. Furthermore, by Sullivan's No-Wandering Domains theorem [Su] every component of the Fatou set $F(f_\omega)$ is eventually periodic.

Since, for every ω, f_ω has essentially one forward orbit of critical points, there cannot exist two or more cycles of periodic components of $F(f_\omega)$ unless all of them are Siegel disks.

In this last case (that is, existence of several cycles of Siegel disks), the critical points lie in the Julia set. By the Riemann-Hurwitz formula we conclude that every component of $F(f_\omega)$ is simply connected.

Assume now that f_ω has a unique cycle of Fatou components which is of attractive type. This cycle contains at least one critical point. Since f_ω has exactly two critical points and takes one to the other, both critical points are contained in this cycle.

If the period of the attractive cycle is one (i.e. the cycle is composed of just one fixed attractive component), then the two critical points belong to the corresponding component. By the Riemann-Hurwitz relation we conclude that the Euler characteristic of this component is $-\infty$ and so it has multiple connectivity. Further, the Fatou set coincides with this attractive component.

If the period of the attractive cycle is greater than one, then the two critical points lie in different components of the cycle. By standard arguments (using the Riemann-Hurwitz relation) one can prove that the Euler characteristic of every component of $F(f_\omega)$ is one. Thus they are all simply connected.

Finally, if f_ω has a unique cycle of Fatou components which is of parabolic type, then by exactly the same arguments as before we conclude that every component of $F(f_\omega)$ is simply connected, unless $F(f_\omega)$ consists of a unique fixed parabolic component, in which case $\chi(F(f_\omega)) = -\infty$.

A fixed attractive component has a corresponding attractive fixed point. Associated to a fixed parabolic component we have a fixed point with derivative equal to one (i.e. a multiple parabolic fixed point). For our family $\mathcal{F}_d$ the second derivative of a multiple parabolic fixed point is non-zero.

So $J(f_\omega)$ is connected if and only if f_ω has neither an attractive fixed point nor a multiple parabolic fixed point. The parameter $\omega_0 = -d(1+1/d)^{d+1}$ is the unique one where f_ω has a multiple parabolic fixed point. The analytic curve $\{\omega(u) = du^{-1}(1-u/d)^{d+1}\ : u \in S^1\ \}$ in the ω - plane is the set of parameters where f_ω has a neutral fixed point. In fact this is a Jordan curve since $\omega(z^{d+1}) = c(z)^{d+1}$ where $c : S^1 \to \mathbf{C}$, $c(z) = \sqrt[d+1]{d}z^{-1}(1 - z^{d+1}/d)$, is a one-to-one map. The parameter ω_0 is the unique singular point in this curve. Further, this Jordan curve separates the ω-plane into two components. The unbounded component coincides with the set of parameters where f_ω has an attractive fixed point.

Putting all this together gives Theorem 2.

We now proceed to point out an equivalent statement for the HD-conjecture that follows from Theorem 1.

The ω-plane is a disjoint union of the sets:
$\mathcal{H} = \{ \omega : f_\omega$ has an attracting periodic point $\}$,
$\mathcal{N} = \{ \omega : f_\omega$ has a neutral periodic point $\}$,
$\mathcal{R} = \{ \omega :$ every periodic point of f_ω is repelling $\}$,
Since the two critical points of f_ω have the same forward orbit, f_ω is hyperbolic if and only if $\omega \in \mathcal{H}$.

The set of parameters for which $J(f_\omega) = \bar{\mathbf{C}}$ is contained in $\mathcal{N} \cup \mathcal{R}$. So, if the HD-conjecture for $\mathcal{F}_d$ is true, then the set $\{ \omega : J(f_\omega) = \bar{\mathbf{C}} \}$ has an empty interior.

Reciprocally, for parameters in $\mathcal{H}$, f_ω is J-stable, while for parameters in $\mathcal{N}$, f_ω is J-unstable (see [MSS]). By Theorem 1 and the classification of Fatou components, $\mathcal{R}$ is contained in $\{ \omega : J(f_\omega) = \bar{\mathbf{C}} \}$. Now if $\{ \omega : J(f_\omega) = \bar{\mathbf{C}} \}$ has an empty interior, then it is contained in the J-unstable parameters. We then conclude that, in this case, $\mathcal{H}$ coincides with the set of J-stable parameters.

Since by [MSS] we know in addition that the set of J-stable parameters is dense in $\mathbf{C}$, we conclude that if $\{ \omega : J(f_\omega) = \bar{\mathbf{C}} \}$ has an empty interior, then the HD-conjecture for $\mathcal{F}_d$ holds.

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