GRADIENT ESTIMATES FOR POSITIVE SOLUTIONS OF THE LAPLACIAN WITH DRIFT

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Abstract. Let \( M \) be a complete Riemannian manifold of dimension \( n \) without boundary and with Ricci curvature bounded below by \(-K\), where \( K \geq 0 \). If \( b \) is a vector field such that \( \|b\| \leq \gamma \) and \( \nabla b \leq K* \) on \( M \), for some nonnegative constants \( \gamma \) and \( K* \), then we show that any positive \( C^\infty(M) \) solution of the equation \( \Delta u(x) + (b(x)|\nabla u(x)) = 0 \) satisfies the estimate
\[
\frac{\|\nabla u\|^2}{u^2} \leq \frac{n(K + K*)}{w} + \frac{\gamma^2}{w(1 - w)} ,
\]
on \( M \), for all \( w \in (0,1) \). In particular, for the case when \( K = K* = 0 \), this estimate is advantageous for small values of \( \|b\| \) and when \( b \equiv 0 \) it recovers the celebrated Liouville theorem of Yau (Comm. Pure Appl. Math. 28 (1975), 201–228).

1. Introduction

In this paper we investigate the behaviour of positive \( C^\infty(M) \) solutions of the equation
\[
(1.1) \quad \Delta u(x) + (b(x)|\nabla u(x)) = 0
\]
on \( M \), where \( M \) is an \( n \)-dimensional complete Riemannian manifold without boundary.

We require smoothness of the manifold, uniform bound on the norm of the vector field \( b \) as well as lower bounds on the tensor fields of the Ricci curvature and \( \nabla b \) where
\[
(1.2) \quad \nabla b(X,Y) = \langle \nabla_X b, Y \rangle , \quad \forall X,Y \in \mathfrak{X}(M) ,
\]
where \( \mathfrak{X}(M) \) denotes the Lie algebra of vectors fields on \( M \) and \( \nabla_X b \) the associated (Levi-Civita) Riemannian covariant derivative of \( b \) with respect to \( X \).

Our main result is a gradient estimate for positive \( C^\infty(M) \) solutions of equation (1.1), namely,
\[
(1.3) \quad \frac{\|\nabla u\|^2}{u^2} \leq \frac{n(K + K*)}{w} + \frac{\gamma^2}{w(1 - w)} ,
\]
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on $M$, for any $w \in (0, 1)$, where the Ricci curvature is bounded below by $-K$, $\nabla b \leq K$, and $\|b\| \leq \gamma$ for some nonnegative constants $K$, $K_*$ and $\gamma$.

For the particular case when $K = K_* = 0$ inequality (1.3) yields

$$\|\nabla u\|^2 \leq 4\gamma^2.$$

Note that this simple estimation is independent of the dimension of $M$ and for the case when $b \equiv 0$ it recovers the Liouville theorem of Yau [10]. The proof of (1.3), and thus of (1.4), is essentially along the lines of Li and Yau [7] and Davies [3, Chap. 5].

This method, originated first in Yau [10] and Cheng and Yau [2], has been developed by several authors (cf. [3], [6], [7], [8], and [9], amongst others). More specifically, for the case when $b = \nabla \phi$, and $\phi \in C^\infty(M)$, a gradient estimate for any positive $C^\infty(M)$ solution of (1.1) has been obtained by Setti in [9].

In order to start, however, we need an extension of the Bochner-Lichnérówicz-Weitzenböck formula for the operator $L^b = \Delta + (b\nabla )$. This remarkable fact is proved as an independent lemma. It is known for drift vectors $b = \nabla \phi$ of gradient form; see e.g. the monograph of Deuschel and Stroock (cf. [4], §6.2).

2. Gradient estimates revisited

In the derivation of the main results, a central role will be played by the next formula.

**Lemma 2.1** (Bochner-Lichnérówicz-Weitzenböck formula for $L^b$). Let $M$ be a Riemannian manifold and assume that $f \in C^\infty(M)$. Then,

\[ L^b(\|\nabla f\|^2) = 2\|\text{Hess}(f)\|_{h.s.}^2 + 2(\nabla f|\nabla(\nabla^b f)) + 2(\text{Ric} - \nabla b)(\nabla f, \nabla f), \]

where $\|\text{Hess}(f)\|_{h.s.}$ denotes the Hilbert-Schmidt norm of Hess$(f)$ (cf. [4, p. 262]), Ric denotes the Ricci curvature and $\nabla b$ denotes the tensor field given by (1.2).

**Proof.** Applying the well-known Bochner-Lichnérówicz-Weitzenböck formula for the Laplacian, one obtains

\[ L^b(\|\nabla f\|^2) = 2\|\text{Hess}(f)\|_{h.s.}^2 + 2(\nabla f|\nabla(\Delta f)) + 2\text{Ric}(\nabla f, \nabla f) + (b|\nabla(\|\nabla f\|^2)). \]

So, in order to prove (2.1) first we establish that

\[ \nabla b(\nabla f, \nabla f) = (\nabla((b|\nabla f))(\nabla f)) - \frac{1}{2}(b|\nabla(\|\nabla f\|^2)). \]

Now, one has

\[ \nabla b(\nabla f, \nabla f) = (\nabla_{\nabla f} b|\nabla f) = \nabla_{\nabla f}((b|\nabla f)) - (b|\nabla_{\nabla f} \nabla f) \]

\[ = -(b|\nabla_{\nabla f} \nabla f) + \nabla f((b|\nabla f)) = -(b|\nabla_{\nabla f} \nabla f) + (\nabla((b|\nabla f))|\nabla f). \]

Thus, all that remains is to show that

\[ \nabla_{\nabla f} \nabla f = \frac{1}{2}(\nabla(\|\nabla f\|^2)). \]

In order to check (2.3) let any $Z \in \mathcal{X}(M)$; then

\[ (\nabla((\|\nabla f\|^2)), Z) = Z((\|\nabla f\|^2)) = Z((\nabla f|\nabla f)) \]

\[ = 2(\nabla Z|\nabla f)(\nabla f) = 2(\nabla_{\nabla f} Z + [Z, \nabla f]|\nabla f) \]

\[ = 2(\nabla_{\nabla f} Z|\nabla f) + 2([Z, \nabla f]|\nabla f) \]
(2.1) is written as

\[ \Delta u(x, t) + (b(x)|\nabla u(x, t)| = \frac{\partial u(x, t)}{\partial t}, \]

on \( M \times [0, \infty) \), then for any \( \alpha > 1 \) and any \( w \in (0, 1) \), the estimate

\[
\frac{\| \nabla u \|^2 (x, t)}{u(x, t)} - \alpha \frac{u_t(x, t)}{u(x, t)} \leq \frac{\alpha^2}{2w} + \frac{\alpha^2}{2w} \left\{ \frac{2\nu}{R^2} + \frac{(n - 1)(1 + R\sqrt{K})}{R^2} + \frac{\nu}{R^2} + \frac{K + K_*}{2(\alpha - 1)} \right. \\
\left. + \frac{\gamma \epsilon}{R} + \frac{n}{8(1 - w)(\alpha - 1)} \left( \frac{2\gamma}{n} + \frac{\alpha \epsilon}{R} \right)^2 \right\}
\]

(2.5) holds on \( B_p(R) \times (0, \infty) \), where \( \epsilon > 0 \) and \( \nu > 0 \) are some constants.

**Proof.** Observe that the function \( f(x, t) = \log u(x, t) \) satisfies the equation

\[ L^f f + \| \nabla f \|^2 = f_t. \]

Now, using formula (2.1), it follows that

\[ F(x, t) = t\{ \| \nabla f \|^2 (x, t) - \alpha f_t(x, t) \}, \]
satisfies the estimate
\[
L^bf - F_t + 2(\nabla f \mid \nabla F) + t^{-1}F \\
t = t \left[ L^b(\|f\|^2) - \alpha L^bf_t - 2(\nabla f \mid \nabla f_t) + \alpha f_t \\
+ 2(\nabla f \mid \nabla (\|f\|^2)) - 2\alpha(\nabla f \mid \nabla f_t) \right] \\
\geq t \left[ \frac{2(\Delta f)^2}{n} - 2(K + K_*) \|f\|^2 \right] \\
= t \left[ \frac{2}{n} \|f\|^2 + (b \mid \nabla f) - f_t^2 - 2(K + K_*) \|f\|^2 \right]
\]
on \( B_p(2R) \times (0, \infty) \), where we have used the inequalities \( \|\text{Hess } f\|_{\text{H.S.}} \geq (\Delta f)^2/n \) and \( (\text{Ric} - \nabla b) \geq -(K + K_*) \).

Let \( \psi \) be a \( C^\infty(\mathbb{R}) \) function such that
\[
\psi(r) = \begin{cases} 
1 & \text{if } r \in (-\infty, 1], \\
0 & \text{if } r \in [2, \infty), 
\end{cases}
\]and \( 0 \leq \psi(r) \leq 1, \forall r \in \mathbb{R} \).

Denote by \( \epsilon > 0 \) and \( \nu > 0 \) some constants with
\[
0 \geq \psi^{-1/2}(r) \frac{d}{dr} \psi(r) \geq -\epsilon
\]and
\[
\frac{d^2}{dr^2} \psi(r) \geq -\nu.
\]

Now we set \( \phi(x) = \psi \left( \frac{d(p, x)}{R} \right) \), where \( d(p, x) \) is the distance between \( p \) and \( x \).

Using an argument of Calabi [1] (see also Cheng and Yau [2] and Setti [9]), we can assume without loss of generality that the function \( \phi \), with support in \( B_p(2R) \), is of class \( C^2 \).

Let \( (a, s) \) be the point in \( B_p(2R) \times [0, t] \) at which \( \phi F \) takes its maximum value, and assume that this value is positive (otherwise the proof is trivial). Then at \( (a, s) \) one has
\[
\nabla(\phi F) = 0, \quad \Delta(\phi F) \leq 0, \quad F_t \geq 0.
\]
Therefore at \( (a, s) \) one has
\[
\phi \Delta F + F \Delta F - 2 + \|\nabla\phi\|^2\phi^{-1} \leq 0.
\]
This inequality together with the estimates
\[
(2.6) \quad \|\nabla\phi\|^2 \leq \frac{c^2 \phi}{R^2},
\]
and
\[
(2.7) \quad \Delta \phi \geq -\frac{(n - 1)(1 + R\sqrt{K})\epsilon^2}{R^2} - \frac{\nu}{R^2}
\]
(cf. [1]) yields
\[
(2.8) \quad \phi \Delta F \leq \left( \frac{2^2}{R^2} + \frac{(n - 1)(1 + R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right) F, \quad \text{at } (a, s).
\]
Inequalities (2.6) and (2.7) at \( (a, s) \) imply that
\[
\phi \Delta F - (b \mid \nabla \phi) F - \phi F_t - 2(\nabla f \mid \nabla \phi) F + s^{-1} \phi F
\]
From (2.8) the left-hand side of (2.9) satisfies
\[ \phi \Delta F - (b \mid \nabla \phi) F - \phi F_t - 2(\nabla f \mid \nabla \phi) F + s^{-1} \phi F \]
\[ \leq \left( \frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right) F - (b \mid \nabla \phi) F - 2(\nabla f \mid \nabla \phi) F + s^{-1} \phi F. \]

Denoting \( \mu = \| \nabla f \|^2 (a, s) \), using (2.9) and the last inequality, we obtain
\[ \left( \frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right) F \\
+ \frac{\epsilon\gamma s^{1/2} F}{R} + \frac{2(\mu\phi)^{1/2} F^{3/2}}{R} + s^{-1} \phi F \]
\[ \leq \left\{ \frac{2}{n} \left( \mu - \frac{\mu s - 1}{s} \right)^2 F^2 - \frac{4(\mu\phi)^{1/2} F^{1/2}}{n} \right\} \left( \mu - \frac{\mu s - 1}{s} \right) F \phi - 2(K + \lambda_*) s \mu \phi F. \]

Multiplying this inequality by \( s \phi \) and since \( \phi^2 \leq 1 \), we obtain
\[ \frac{2(1 + (\alpha - 1)\mu)s^2(\phi F)^2}{\alpha^2 n} - 2 \left\{ \frac{2s\gamma \mu^{1/2}(1 + (\alpha - 1)\mu)}{\alpha n} + \frac{\epsilon s^{1/2} \mu^{1/2}}{R} \right\} (\phi F)^{3/2} \]
\[ - \left\{ \frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right\} s + 1 \\
+ \frac{\gamma s^2}{R} + 2(K + \lambda_*) \mu s^2 \right\} (\phi F) \leq 0. \]

On the other hand, for any \( w \in (0, 1) \) we have
\[ -2 \left\{ \frac{2s\gamma \mu^{1/2}(1 + (\alpha - 1)\mu)}{\alpha n} + \frac{\epsilon s^{1/2} \mu^{1/2}}{R} \right\} (\phi F)^{3/2} \]
\[ \geq - \frac{2(1-w)(1 + (\alpha - 1)\mu)s^2(\phi F)^2}{\alpha^2 n} \]
\[ (2.11) \]
\[ - \frac{n}{2(1-w)(1 + (\alpha - 1)\mu)s^2} \left[ \frac{2s\gamma \mu^{1/2}(1 + (\alpha - 1)\mu)}{n} + \frac{\epsilon s^{1/2} \mu^{1/2}}{R} \right]^2 (\phi F). \]

From (2.11) inequality (2.10) becomes
\[ A_1 \lambda^2 - 2A_2 \lambda \leq 0, \]
where
\[ \lambda = \phi F, \quad A_1 = \frac{2w}{\alpha^2 n}(1 + (\alpha - 1)\mu)s^2, \]
and
\[ 2A_2 = \left( \frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right) s + 1 + \frac{\gamma s^2}{R} + 2(K + \lambda_*) \mu s^2 \\
+ \frac{n}{2(1-w)(1 + (\alpha - 1)\mu)s^2} \left[ \frac{2s\gamma \mu^{1/2}(1 + (\alpha - 1)\mu)}{n} + \frac{\epsilon s^{1/2} \mu^{1/2}}{R} \right]^2. \]
As in [3, Lemma 5.3.3], we use the estimate
\[
\mu s^2 \leq \frac{s}{(1 + (\alpha - 1)\mu s)^2}.
\]
and so we obtain
\[
\frac{2A_2}{A_1} \leq \frac{n\alpha^2}{2w} + \frac{n\alpha^2 s}{2w} \left\{ \frac{2\epsilon^2}{R^2} + \frac{(n - 1)(1 + R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right\} + \frac{K + K_*}{2(\alpha - 1)} + \frac{\gamma \epsilon}{R} + \frac{n}{8(1 - w)(\alpha - 1)} \left( \frac{2\gamma}{w} + \frac{\epsilon \alpha}{w} \right)^2.
\]
(2.12)

Now, since \( \lambda \leq \frac{2A_2}{A_1} \), \( s \in [0, t] \) and using (2.12), estimate (2.5) holds.

From Theorem 2.1 one obtains the next global gradient estimate

**Corollary 2.1.** Let \( M \) be a complete Riemannian manifold of dimension \( n \) without boundary and assume that the Ricci curvature of \( M \) is bounded from below by \( -K \) with \( K \geq 0 \). Also we suppose that the vector field \( b \) satisfies \( \|b\| \leq \gamma \) and that the tensor field \( \nabla b \), given by (1.2), is bounded from above by \( K_* \), for some nonnegative constants \( \gamma \) and \( K_* \). If \( u(x) \) is a positive \( C^\infty(M) \) solution of equation (1.1), then for any \( w \in (0, 1) \), the following estimate holds on \( M \):

\[
\frac{\|\nabla u\|^2}{u^2} \leq \frac{n(K + K_*)}{w} + \frac{\gamma^2}{w(1 - w)}.
\]

**Proof.** Letting \( R \to \infty \) and \( t \to \infty \) in (2.5) one has

\[
\frac{\|\nabla u\|^2}{u^2} \leq \frac{n\alpha^2(K + K_*)}{4w(\alpha - 1)} + \frac{\alpha^2\gamma^2}{4w(1 - w)(\alpha - 1)},
\]

on \( M \). Setting \( \alpha = 2 \) (which minimizes the right-hand side of (2.13)), the result holds.

**Remark 2.2.** If \( u(x) \) is a positive \( C^\infty(M) \) solution of \( \Delta u(x) + (b(x)|\nabla u(x)) = 0 \), and assuming that \( \text{Ric} \geq 0, \nabla b \leq 0 \) and \( \|b\| \leq \gamma \), for some \( \gamma \geq 0 \), it follows from Corollary 2.1 above that

\[
\frac{\|\nabla u\|^2}{u^2} \leq \frac{\gamma^2}{w(1 - w)},
\]

for any \( w \in (0, 1) \). Setting \( w = 1/2 \) (which minimizes the right-hand side of (2.14)) one obtains

\[
\frac{\|\nabla u\|^2}{u^2} \leq 4\gamma^2.
\]

**Remark 2.3.** Let \( M = \mathbb{R} \) be the one-dimensional Euclidean space with its standard Riemannian metric. It is a complete Riemannian manifold without boundary and with Ricci curvature identically zero. In this setting consider the equation

\[
u''(x) + bu'(x) = 0,
\]

where \( b \) is a real constant. It is clear that \( u(x) = e^{-bx} \) is a positive \( C^\infty(\mathbb{R}) \) solution of (2.15), such that \( \frac{\|\nabla u\|^2}{u^2} = b^2 \). This case is contemplated by Corollary 2.1 with \( K = K_* = 0 \) and \( \gamma = |b| \), which establishes that \( \frac{\|\nabla u\|^2}{u^2} \leq 4b^2 \).
On the other hand, the equation

\[ u''(x) - (1 + e^x)u'(x) = 0 \]  

has the function \( u(x) = e^{e^x} \) as a positive \( C^\infty(\mathbb{R}) \) solution such that \( \frac{\|\nabla u\|^2}{u^2} \) is unbounded. Note that the function \( b(x) = -(1 + e^x) \) satisfies \( b' \leq 0 \) and \( b \) is unbounded. Thus, we see that the assumption of the boundedness of \( \|b\| \) is needed for the kind of results obtained here.

References


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