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GRADIENT ESTIMATES FOR POSITIVE SOLUTIONS OF THE LAPLACIAN WITH DRIFT

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ABSTRACT. Let M be a complete Riemannian manifold of dimension n without boundary and with Ricci curvature bounded below by $-K$, where $K \geq 0$. If b is a vector field such that $\|b\| \leq \gamma$ and $\nabla b \leq K_*$ on M, for some nonnegative constants γ and K_*, then we show that any positive $\mathcal{C}^\infty(M)$ solution of the equation $\Delta u(x) + (b(x)|\nabla u(x)) = 0$ satisfies the estimate

$$\frac{\|\nabla u\|^2}{u^2} \leq \frac{n(K+K_*)}{w} + \frac{\gamma^2}{w(1-w)} ,$$

on M, for all $w \in (0,1)$. In particular, for the case when $K = K_* = 0$, this estimate is advantageous for small values of $\|b\|$ and when $b \equiv 0$ it recovers the celebrated Liouville theorem of Yau (*Comm. Pure Appl. Math.* **28** (1975), 201–228).

1. Introduction

In this paper we investigate the behaviour of positive $\mathcal{C}^\infty(M)$ solutions of the equation

$$\Delta u(x) + (b(x)|\nabla u(x)) = 0 \tag{1.1}$$

on M, where M is an n-dimensional complete Riemannian manifold without boundary.

We require smoothness of the manifold, uniform bound on the norm of the vector field b as well as lower bounds on the tensor fields of the Ricci curvature and ∇b where

$$\nabla b(X,Y) = (\nabla_X b|Y), \quad \forall X, Y \in \mathfrak{X}(M), \tag{1.2}$$

where $\mathfrak{X}(M)$ denotes the Lie algebra of vectors fields on M and $\nabla_X b$ the associated (Levi-Civita) Riemannian covariant derivative of b with respect to X.

Our main result is a gradient estimate for positive $\mathcal{C}^\infty(M)$ solutions of equation (1.1), namely,

$$\frac{\|\nabla u\|^2}{u^2} \leq \frac{n(K+K_*)}{w} + \frac{\gamma^2}{w(1-w)} , \tag{1.3}$$

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on M, for any $w \in (0,1)$, where the Ricci curvature is bounded below by $-K$, $\nabla b \leq K_*$ and $\|b\| \leq \gamma$ for some nonnegative constants K, K_* and γ.

For the particular case when $K = K_* = 0$ inequality (1.3) yields

$$\frac{\|\nabla u\|^2}{u^2} \leq 4\gamma^2. \tag{1.4}$$

Note that this simple estimation is independent of the dimension of M and for the case when $b \equiv 0$ it recovers the Liouville theorem of Yau [10]. The proof of (1.3), and thus of (1.4), is essentially along the lines of Li and Yau [7] and Davies [3, Chap. 5].

This method, originated first in Yau [10] and Cheng and Yau [2], has been developed by several authors (cf. [3], [6], [7], [8], and [9], amongst others). More specifically, for the case when $b = \nabla\phi$, and $\phi \in \mathcal{C}^\infty(M)$, a gradient estimate for any positive $\mathcal{C}^\infty(M)$ solution of (1.1) has been obtained by Setti in [9].

In order to start, however, we need an extension of the Bochner-Lichnèrowicz-Weitzenböck formula for the operator $L^b = \Delta + (b|\nabla\ )$. This remarkable fact is proved as an independent lemma. It is known for drift vectors $b = \nabla\phi$ of gradient form; see e.g. the monograph of Deuschel and Stroock (cf. [4], §6.2).

2. Gradient estimates revisited

In the derivation of the main results, a central role will be played by the next formula.

Lemma 2.1 (Bochner-Lichnèrowicz-Weitzenböck formula for L^b)**.** *Let M be a Riemannian manifold and assume that $f \in \mathcal{C}^\infty(M)$. Then,*

$$L^b(\| \nabla f \|^2) = 2\| \mathit{Hess}(f) \|^2_{H.S.} + 2(\nabla f|\nabla(L^b f)) + 2(\ \mathit{Ric} - \nabla b)(\nabla f, \nabla f), \tag{2.1}$$

where $\| \mathit{Hess}(f) \|_{H.S.}$ denotes the Hilbert-Schmidt norm of Hess(f) (cf. [4, p. 262]*), Ric denotes the Ricci curvature and ∇b denotes the tensor field given by* (1.2)*.*

Proof. Applying the well-known Bochner-Lichnèrowicz-Weitzenböck formula for the Laplacian, one obtains

$$L^b(\| \nabla f \|^2) = 2\| \mathrm{Hess}(f) \|^2_{\mathrm{H.S.}} + 2(\nabla f|\nabla(\Delta f)) + 2\,\mathrm{Ric}(\nabla f, \nabla f) + (b|\nabla(\| \nabla f \|^2)).$$

So, in order to prove (2.1) first we establish that

$$\nabla b(\nabla f, \nabla f) = (\nabla((b|\nabla f))|\nabla f) - \frac{1}{2}(b|\nabla(\| \nabla f \|^2)). \tag{2.2}$$

Now, one has

$$\begin{aligned}\nabla b(\nabla f, \nabla f) &= (\nabla_{\nabla f} b|\nabla f) = \nabla_{\nabla f}((b|\nabla f)) - (b|\nabla_{\nabla f}\nabla f)\\ &= -(b|\nabla_{\nabla f}\nabla f) + \nabla f((b|\nabla f)) = -(b|\nabla_{\nabla f}\nabla f) + (\nabla((b|\nabla f))|\nabla f).\end{aligned}$$

Thus, all that remains is to show that

$$\nabla_{\nabla f}\nabla f = \frac{1}{2}\nabla(\| \nabla f \|^2). \tag{2.3}$$

In order to check (2.3) let any $Z \in \mathfrak{X}(M)$; then

$$\begin{aligned}(\nabla(\| \nabla f \|^2), Z) &= Z(\| \nabla f \|^2) = Z((\nabla f|\nabla f))\\ &= 2(\nabla_Z \nabla f|\nabla f) = 2((\nabla_{\nabla f} Z + [Z, \nabla f])|\nabla f)\\ &= 2(\nabla_{\nabla f} Z|\nabla f) + 2([Z, \nabla f]|\nabla f)\end{aligned}$$

$$= 2\{\nabla f((Z|\nabla f)) - (Z|\nabla_{\nabla f})\} + 2([Z, \nabla f]|\nabla f)$$
$$= 2\{\nabla f(Zf) - (Z|\nabla_{\nabla f}\nabla f)\} + 2([Z, \nabla f]|\nabla f)$$
$$= 2\{[\nabla f, Z]f + Z((\nabla f)f) - (Z|\nabla_{\nabla f})\} + 2([Z, \nabla f]|\nabla f)$$
$$= -2(Z|\nabla_{\nabla f}\nabla f) + 2Z(\| \nabla f \|^2),$$

where, for any $X, Y \in \mathfrak{X}(M)$, $[X, Y] \equiv XY - YX$ is the commutator of X and Y. Therefore,

$$(\nabla(\| \nabla f \|^2)|Z) = -2(Z|\nabla_{\nabla f}\nabla f) + 2(\nabla(\| \nabla f \|^2)|Z),$$

and so

$$(\nabla(\| \nabla f \|^2)|Z) = 2(\nabla_{\nabla f}\nabla f|Z), \quad \forall Z \in \mathfrak{X}(M),$$

from which (2.3) follows. □

Remark 2.1. For the case when $b = -\nabla U$, $U \in \mathcal{C}^\infty(M)$, denoting $L^U = L^{-\nabla U}$ and taking into account that $Hess(U)(X, Y) = (\nabla_X(\nabla U)|Y)$, for all $X, Y \in \mathfrak{X}(M)$ (cf. [4, p. 261]), formula (2.1) is written as

$$L^U(\| \nabla f \|^2) = 2\| Hess(f) \|^2_{\text{H.S.}} + 2(\nabla f|\nabla(L^U f)) + 2(Ric + Hess(U))(\nabla f, \nabla f),$$

which agrees with the Bochner-Lichnèrowicz-Weitzenböck formula derived in [4, p. 262] and [5, p. 32].

Now, formula (2.1) enables us to prove the next local gradient estimate.

Theorem 2.1. *Let M be a complete Riemannian manifold of dimension n without boundary. Let $B_p(2R)$ be a geodesic ball of radius $2R$ around $p \in M$ and denote by $-K(2R)$, with $K(2R) \geq 0$, a lower bound on $B_p(2R)$ of the Ricci curvature. Set b a vector field on M and denote by $\gamma(2R)$ and $K_*(2R)$ some nonnegative constants satisfying $\| b \| \leq \gamma(2R)$ and $\nabla b \leq K_*(2R)$ on $B_p(2R)$, where ∇b is the tensor field given by (1.2). If $u(x, t)$ is a positive $\mathcal{C}^\infty$ solution of the equation*

$$\Delta u(x, t) + (b(x)|\nabla u(x, t)) = \frac{\partial u(x, t)}{\partial t}, \tag{2.4}$$

on $M \times [0, \infty)$, then for any $\alpha > 1$ and any $w \in (0, 1)$, the estimate

$$\frac{\| \nabla u \|^2 (x, t)}{u^2(x, t)} - \alpha \frac{u_t(x, t)}{u(x, t)}$$
$$\leq \frac{n\alpha^2}{2wt} + \frac{n\alpha^2}{2w}\left\{\frac{2\epsilon^2}{R^2} + \frac{(n-1)(1 + R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} + \frac{K + K_*}{2(\alpha - 1)}\right.$$
$$\left. + \frac{\gamma\epsilon}{R} + \frac{n}{8(1-w)(\alpha-1)}\left(\frac{2\gamma}{n} + \frac{\alpha\epsilon}{R}\right)^2\right\} \tag{2.5}$$

holds on $B_p(R) \times (0, \infty)$, where $\epsilon > 0$ and $\nu > 0$ are some constants.

Proof. Observe that the function $f(x, t) = \log\ u(x, t)$ satisfies the equation

$$L^b f + \| \nabla f \|^2 = f_t.$$

Now, using formula (2.1), it follows that

$$F(x, t) = t\{\| \nabla f \|^2\ (x, t) - \alpha f_t(x, t)\},$$

satisfies the estimate

$$\begin{aligned} L^bF - F_t + 2(\nabla f \mid \nabla F) + t^{-1}F &= t\left[L^b(\| \nabla f \|^2) - \alpha L^b f_t - 2(\nabla f \mid \nabla f_t) + \alpha f_{tt} \right. \\ &\quad \left. +2(\nabla f \mid \nabla(\| \nabla f \|^2)) - 2\alpha(\nabla f \mid \nabla f_t)\right] \\ &\geq t\left[\frac{2(\Delta f)^2}{n} - 2(K+K_*) \| \nabla f \|^2\right] \\ &= t\left[\frac{2}{n}\{\| \nabla f \|^2 + (b \mid \nabla f) - f_t\}^2 - 2(K+K_*) \| \nabla f \|^2\right] \end{aligned}$$

on $B_p(2R) \times (0,\infty)$, where we have used the inequalities $\|\text{Hess } f \|^2_{\text{H.S.}} \geq (\Delta f)^2/n$ and $(\text{Ric } -\nabla b) \geq -(K+K_*)$.

Let ψ be a $\mathcal{C}^\infty(\mathbb{R})$ function such that

$$\psi(r) = \begin{cases} 1 & \text{if} \quad r \in (-\infty, 1], \\ 0 & \text{if} \quad r \in [2,\infty), \end{cases}$$

and $0 \leq \psi(r) \leq 1, \ \ \forall r \in \mathbb{R}$.

Denote by $\epsilon > 0$ and $\nu > 0$ some constants with

$$0 \geq \psi^{-1/2}(r)\frac{d}{dr}\psi(r) \geq -\epsilon$$

and

$$\frac{d^2}{dr^2}\psi(r) \geq -\nu.$$

Now we set $\phi(x) = \psi\left(\frac{d(p,x)}{R}\right)$, where $d(p,x)$ is the distance between p and x. Using an argument of Calabi [1] (see also Cheng and Yau [2] and Setti [9]), we can assume without loss of generality that the function ϕ, with support in $B_p(2R)$, is of class $\mathcal{C}^2$.

Let (a,s) be the point in $B_p(2R) \times [0,t]$ at which ϕF takes its maximum value, and assume that this value is positive (otherwise the proof is trivial). Then at (a,s) one has

$$\nabla(\phi F) = 0, \quad \Delta(\phi F) \leq 0, \quad F_t \geq 0.$$

Therefore at (a,s) one has

$$\phi \Delta F + F\Delta\phi - 2F\| \nabla \phi \|^2 \phi^{-1} \leq 0.$$

This inequality together with the estimates

$$\| \nabla \phi \|^2 \leq \frac{\epsilon^2 \phi}{R^2}, \tag{2.6}$$

and

$$\Delta\phi \geq -\frac{(n-1)(1+R\sqrt{K})\epsilon^2}{R^2} - \frac{\nu}{R^2} \tag{2.7}$$

(cf. [1]) yields

$$\phi\Delta F \leq \left(\frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2}\right)F, \quad \text{at } (a,s). \tag{2.8}$$

Inequalities (2.6) and (2.7) at (a,s) imply that

$$\phi\Delta F - (b|\nabla\phi)F - \phi F_t - 2(\nabla f|\nabla\phi)F + s^{-1}\phi F$$

$$\geq \left\{ \frac{2}{n} \left[\| \nabla f \|^2 + (b \mid \nabla f) - f_t \right]^2 - 2(K+K_*) \| \nabla f \|^2 \right\} s\phi. \tag{2.9}$$

From (2.8) the left-hand side of (2.9) satisfies

$$\phi \Delta F - (b \mid \nabla\phi)F - \phi F_t - 2(\nabla f \mid \nabla \phi)F + s^{-1}\phi F$$
$$\leq \left(\frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right) F - (b \mid \nabla\phi)F - 2(\nabla f \mid \nabla\phi)F + s^{-1}\phi F.$$

Denoting $\mu = \dfrac{\| \nabla f \|^2 (a,s)}{F(a,s)}$, using (2.9) and the last inequality, we obtain

$$\left(\frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right) F$$
$$+ \frac{\epsilon\gamma\phi^{1/2}F}{R} + \frac{2(\mu\phi)^{1/2}\epsilon F^{3/2}}{R} + s^{-1}\phi F$$
$$\geq \left\{ \frac{2}{n} \left(\mu - \frac{\mu s - 1}{\alpha s} \right)^2 F^2 - \frac{4(\mu F)^{1/2}\gamma}{n} \left(\mu - \frac{\mu s - 1}{\alpha s} \right) F \right\} s\phi - 2\left(K+K_*\right) s\mu\phi F.$$

Multiplying this inequality by $s\phi$ and since $\phi^2 \leq 1$, we obtain

$$\begin{aligned} &\frac{2(1+(\alpha-1)\mu s)^2(\phi F)^2}{\alpha^2 n} - 2\left\{ \frac{2s\gamma\mu^{1/2}(1+(\alpha-1)\mu s)}{\alpha n} + \frac{\epsilon s \mu^{1/2}}{R} \right\} (\phi F)^{3/2} \\ &\quad - \left\{ \left(\frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})R^2}{\epsilon} + \frac{\nu}{R^2} \right) s + 1 \right. \\ &\qquad \left. + \frac{\gamma\epsilon s}{R} + 2\left(K+K_*\right)\mu s^2 \right\} (\phi F) \leq 0. \end{aligned} \tag{2.10}$$

On the other hand, for any $w \in (0,1)$ we have

$$\begin{aligned} &-2\left\{ \frac{2s\gamma\mu^{1/2}(1+(\alpha-1)\mu s)}{\alpha n} + \frac{\epsilon s\mu^{1/2}}{R} \right\} (\phi F)^{3/2} \\ &\geq - \frac{2(1-w)(1+(\alpha-1)\mu s)^2(\phi F)^2}{\alpha^2 n} \\ &- \frac{n}{2(1-w)(1+(\alpha-1)\mu s)^2} \left[\frac{2s\gamma\mu^{1/2}(1+(\alpha-1)\mu s)}{n} + \frac{\epsilon s\alpha\mu^{1/2}}{R} \right]^2 (\phi F). \end{aligned} \tag{2.11}$$

From (2.11) inequality (2.10) becomes

$$A_1\lambda^2 - 2A_2\lambda \leq 0,$$

where

$$\lambda = \phi F, \quad A_1 = \frac{2w}{\alpha^2 n}(1+(\alpha-1)\mu s)^2,$$

and

$$\begin{aligned} 2A_2 = &\left(\frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right) s + 1 + \frac{\gamma\epsilon s}{R} + 2\left(K+K_*\right)\mu s^2 \\ &+ \frac{n}{2(1-w)(1+(\alpha-1)\mu s)^2} \left[\frac{2s\gamma\mu^{1/2}(1+(\alpha-1)\mu s)}{n} + \frac{\epsilon s\alpha\mu^{1/2}}{R} \right]^2. \end{aligned}$$

As in [3, Lemma 5.3.3], we use the estimate

$$\frac{\mu s^2}{(1+(\alpha-1)\mu s)^2} \leq \frac{s}{4(\alpha-1)},$$

and so we obtain

$$\frac{2A_2}{A_1} \leq \frac{n\alpha^2}{2w} + \frac{n\alpha^2 s}{2w}\left\{\left(\frac{2\epsilon^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2}\right)\right.$$

$$\left. + \frac{K+K_*}{2(\alpha-1)} + \frac{\gamma\epsilon}{R} + \frac{n}{8(1-w)(\alpha-1)}\left(\frac{2\gamma}{n} + \frac{\epsilon\alpha}{R}\right)^2\right\}. \tag{2.12}$$

Now, since $\lambda \leq 2A_2/A_1$, $s \in [0,t]$ and using (2.12), estimate (2.5) holds. □

From Theorem 2.1 one obtains the next global gradient estimate

Corollary 2.1. *Let M be a complete Riemannian manifold of dimension n without boundary and assume that the Ricci curvature of M is bounded from below by $-K$ with $K \geq 0$. Also we suppose that the vector field b satisfies $\|b\| \leq \gamma$ and that the tensor field ∇b, given by (1.2), is bounded from above by K_* , for some nonnegative constants γ and K_*. If $u(x)$ is a positive $\mathcal{C}^\infty(M)$ solution of equation (1.1), then for any $w \in (0,1)$, the following estimate holds on M:*

$$\frac{\|\nabla u\|^2}{u^2} \leq \frac{n(K+K_*)}{w} + \frac{\gamma^2}{w(1-w)}.$$

Proof. Letting $R \to \infty$ and $t \to \infty$ in (2.5) one has

$$\frac{\|\nabla u\|^2}{u^2} \leq \frac{n\alpha^2(K+K_*)}{4w(\alpha-1)} + \frac{\alpha^2\gamma^2}{4w(1-w)(\alpha-1)}\ , \tag{2.13}$$

on M. Setting $\alpha = 2$ (which minimizes the right-hand side of (2.13)), the result holds. □

Remark 2.2. If $u(x)$ is a positive $\mathcal{C}^\infty(M)$ solution of $\Delta u(x) + (b(x) \mid \nabla u(x)) = 0$, and assuming that $\mathrm{Ric} \geq 0, \nabla b \leq 0$ and $\| b \| \leq \gamma$, for some $\gamma \geq 0$, it follows from Corollary 2.1 above that

$$\frac{\|\nabla u\|^2}{u^2} \leq \frac{\gamma^2}{w(1-w)}\ , \tag{2.14}$$

for any $w \in (0,1)$. Setting $w = 1/2$ (which minimizes the right-hand side of (2.14)) one obtains

$$\frac{\|\nabla u\|^2}{u^2} \leq 4\gamma^2.$$

Remark 2.3. Let $M = \mathbb{R}$ be the one-dimensional Euclidean space with its standard Riemannian metric. It is a complete Riemannian manifold without boundary and with Ricci curvature identically zero. In this setting consider the equation

$$u''(x) + bu'(x) = 0, \tag{2.15}$$

where b is a real constant. It is clear that $u(x) = e^{-bx}$ is a positive $\mathcal{C}^\infty(\mathbb{R})$ solution of (2.15), such that $\dfrac{\|\nabla u\|^2}{u^2} = b^2$. This case is contemplated by Corollary 2.1 with $K = K_* = 0$ and $\gamma = |b|$, which establishes that $\dfrac{\|\nabla u\|^2}{u^2} \leq 4b^2$.

On the other hand, the equation

$$u''(x) - (1 + e^x)u'(x) = 0 \tag{2.16}$$

has the function $u(x) = e^{e^x}$ as a positive $\mathcal{C}^\infty(\mathbb{R})$ solution such that $\dfrac{\|\nabla u\|^2}{u^2}$ is unbounded. Note that the function $b(x) = -(1 + e^x)$ satisfies $b' \le 0$ and b is unbounded. Thus, we see that the assumption of the boundedness of $\|b\|$ is needed for the kind of results obtained here.

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