PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 2, February 1999, Pages 437–446
S 0002-9939(99)04585-2

BEHAVIOUR OF HOLOMORPHIC AUTOMORPHISMS ON EQUICONTINUOUS SUBSETS OF THE SPACE $\mathcal{C}(\Omega, E)$

J. M. ISIDRO

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Consider a compact Hausdorff topological space Ω, a JB*-triple E and $F := \mathcal{C}(\Omega, E)$, the JB*-triple of all continuous E-valued functions $f : \Omega \to E$ with the pointwise operations and the norm of the supremum. Let G be the group of all holomorphic automorphisms of the unit ball B_F of F that map every equicontinuous subset lying strictly inside B_F into another such a set. The real Banach-Lie group G and its Lie algebra are investigated. The identity connected component of G is identified when E has the strong Banach-Stone property. This extends to the infinite dimensional setting a well known result concerning the case $E = \mathbb{C}$.

1. INTRODUCTION

Let Δ and D denote respectively the unit ball of the complex line $\mathbb{C}$ and the unit ball of $F := \mathcal{C}(\Omega)$, the algebra of all continuous complex valued functions $f : \Omega \to \mathbb{C}$ on the compact Hausdorff space Ω with the norm of the supremum. Let $\mathsf{Aut}(\Delta)$ and $\mathsf{Aut}(D)$ be the corresponding groups of holomorphic automorphisms. The former group is connected and if $\mathsf{Aut}^0(D)$ is the identity connected component of the latter, then a classical result states that $\mathsf{Aut}^0(D)$ and $\mathcal{C}(\Omega, \mathsf{Aut}(\Delta))$ are isomorphic real Lie groups ([4], remark 3.10).

In recent years, much attention has been given to the study of groups of holomorphic automorphisms of bounded domains in infinite dimensional complex Banach spaces, particularly in the case of JB*-triples. These are Banach spaces E whose open unit balls B_E are homogeneous under the action of the group $\mathsf{Aut}(B_E)$ of all holomorphic automorphisms of B_E. Thus it is reasonable to ask whether the above result is valid if we replace $\mathbb{C}$ by an arbitrary JB*-triple E and $\mathcal{C}(\Omega)$ by $F := \mathcal{C}(\Omega, E)$. In this note, the infinite dimensional analogue of that result is obtained. Let G^0 denote the identity connected component of G whenever the latter is a topological group. Here we introduce a closed subgroup $\mathsf{Eaut}(B_F)$ of $\mathsf{Aut}(B_F)$ such that $\mathsf{Eaut}^0(B_F)$ and $\mathcal{C}(\Omega, \mathsf{Aut}^0(B_E))$ are isomorphic as real Banach-Lie groups and $\mathsf{Eaut}B(_F)$ coincides with $\mathsf{Aut}(B_F)$ if $\dim E < \infty$. It turns out that $\mathsf{Eaut}^0(B_F)$ is characterized by the behaviour of its elements in the family $\mathfrak{E}$ of the equicontinuous subsets of B_F that are bounded away from the boundary ∂B_F, a family whose elements are defined without any reference to the holomorphic structure of E or F. Notice also that $\mathfrak{E}$ consists of all relatively compact sets of B_F if $\dim E < \infty$.

Received by the editors November 7, 1996 and, in revised form, May 19, 1997.
1991 *Mathematics Subject Classification.* Primary 46G20, 22E65.

Notation and preliminary results are introduced in section 1. In section 2, the subgroup $\mathsf{Eaut}(B_F)$ of all $\phi \in \mathsf{Aut}(B_F)$ which preserve $\mathfrak{E}$ is studied. It is proved that $\mathsf{Eaut}(B_F)$ is a real Banach-Lie group whose Banach-Lie algebra consists of the triple derivations of F that preserve the family of all bounded equicontinuous subset of F, and that $\mathsf{Eaut}^0(B_F)$ and $\mathcal{C}(\Omega,\, \mathsf{Aut}^0(B_E))$ are isomorphic as real Banach-Lie groups. In section 3, the case in which E has the strong Banach-Stone property is discussed. Necessary and sufficient conditions are established for $\mathsf{Eaut}(B_F)$ to coincide with $\mathsf{Aut}(B_F)$ and an example is given to show that in general $\mathsf{Eaut}(B_F) \neq \mathsf{Aut}(B_F)$.

Notation and preliminary results. For complex Banach spaces Z, W denote by $\mathcal{L}(Z,W)$ the Banach space of all bounded linear operators $Z \to W$. In the Banach algebra $\mathcal{L}(Z) := \mathcal{L}(Z,Z)$ the group of all invertible elements is denoted by $\mathsf{GL}(Z)$. We let B (or B_Z when a reference to Z is needed) be the open unit ball of Z and $\mathsf{Aut}(B)$ is the group of all holomorphic automorphisms of B. A complex Banach space Z with a continuous mapping $(a,b,c) \mapsto \{abc\}$ from $Z \times Z \times Z$ to Z is called a *JB*-triple* if the following conditions are satisfied for all $a,b,c,d \in Z$, where the operator $a\Box b \in \mathcal{L}(Z)$ is defined by $z \mapsto \{abz\}$ and $[\,,\,]$ is the commutator product:

(i) $\{abc\}$ is symmetric complex linear in a,c and conjugate linear in b.

(ii) $[a\Box b,\, c\Box d] = \{abc\}\Box d - c\Box\{dab\}$.

(iii) $a\Box a$ is hermitian and has spectrum ≥ 0.

(iv) $\|\{aaa\}\| = \|a\|^3$.

JB*-triples may be considered as a generalization of C*-algebras and JB*-algebras (see [6],[8]). For every C*-algebra with product $(a,b) \mapsto ab$ and involution $a \mapsto a^*$ the underlying Banach space is a JB*-triple with respect to $\{abc\} := (ab^*c + cb^*a)/2$. Also for every JB*-algebra with Jordan product $(a,b) \mapsto a \circ b$ we get a JB*-triple with $\{abc\} := (a \circ b^*) \circ c - (c \circ a) \circ b^* + (b^* \circ c) \circ a$. A complex Banach space is a JB*-triple if and only if B is homogeneous under the action of the group $\mathsf{Aut}(B)$ and in that case the triple product is uniquely determined. Let Z be a JB*-triple; then a closed subspace $J \subset Z$ is an ideal if $\{ZJZ\} \subset J$ and Z is a JBW*-triple if Z is a dual Banach space. A *derivation* of Z is an element $\delta \in \mathcal{L}(Z)$ such that $\delta\{zzz\} = \{(\delta z)zz\} + \{z(\delta z)z\} + \{zz(\delta z)\}$, $z \in Z$, and an *automorphism* is an element $\phi \in \mathcal{L}(Z)$ such that $\phi\{zzz\} = \{(\phi z)(\phi z)(\phi z)\}$ for $z \in Z$. This occurs if and only if ϕ is a surjective linear isometry of Z, the set of which is denoted by $\mathsf{Isom}(Z)$. Both $\mathsf{Isom}(Z)$ and $\mathsf{Aut}(B)$ are real Banach-Lie groups whose real Banach-Lie algebras are respectively $\mathsf{isom} Z$ (the set of all derivations of Z) and $\mathsf{aut}(B)$ (the set of all *complete* holomorphic vector fields on B). Each $X \in \mathsf{aut}(B)$ is a polynomial of degree ≤ 2 of the form $X_a(z)\frac{\partial}{\partial z} = \big(a + \delta(z) - \{zaz\}\big)\frac{\partial}{\partial z}$ for some $a \in Z$, where z is the coordinate in Z and $\delta \in \mathsf{isom}(Z)$. We have the topologically direct vector space decomposition $\mathsf{aut}(B) = \mathsf{isom}(Z) \oplus \mathfrak{p}$, where $\mathfrak{p}$, the set of *transvections* of B, consists of all vector fields of the form $p_a(z)\frac{\partial}{\partial z} := (a - \{zaz\})\frac{\partial}{\partial z}$ for $a \in Z$. Similarly we have $\mathsf{Aut}(B) = \mathsf{Isom}(Z) \cdot \mathfrak{P}$ where $\mathfrak{P} := \exp \mathfrak{p}$ is the set of all *Moebius transformations* of B. All Banach-Lie groups and Banach-Lie algebras considered here will be endowed with the topology of *local uniform convergence*. Whenever G is a topological group, G^0 is its identity connected component, and we have

$$\mathsf{Aut}^0(B) = \mathsf{Isom}^0(Z) \cdot \mathfrak{P}. \tag{1.0}$$

A vector field X can be considered as a differential operator acting on holomorphic functions and in that case it is represented by $\tilde{X}$. Recall that $\tilde{X}^0 g := g$, $\tilde{X}^1 g(z) := g'(z)X(z)$ and $\tilde{X}^{n+1} g := \tilde{X}^1(\tilde{X}^n g)$ for $n \geq 1$. If g is defined in $C \subset Z$, we let $\|g\|_C := \sup_{z\in C}\|g(z)\|$ which is written $\|g\|$ if C is the unit ball. Let $C \subset B$ be an open set such that $\mathrm{dist}(C, \partial B) = r > 0$. Then ([11], prop. 5.5)

$$\|\tilde{X}^n g\|_C \le (\frac{n}{r})^n \|X\|^n \|g\|$$

for all $n \in \mathbb{N}$. In particular, for $g := \mathsf{id}$ and z in the ball $C := \{z \in Z \mid \mathrm{dist}(z, \partial B) > r\}$, the exponential series $\phi_t(z) := \exp tX(z) := \sum_0^\infty \frac{t^n}{n!}\tilde{X}^n \mathsf{id}(z)$ is dominated by

$$\sum_0^\infty \frac{n^n}{n!}(\frac{|t|.\|X\|}{r})^n, \tag{1.1}$$

which is convergent for $|t|.\|X\| < r$. Moreover we have

$$\lim_{t\to 0}\frac{1}{t}(\phi_t z - z) = X(z) \tag{1.2}$$

uniformly for $z \in C$ if $|t|.\|X\| < r$, since

$$\begin{aligned}\|\frac{1}{t}\big(\phi_t(z) - z\big) - X(z)\| &= \|\sum_2^\infty \frac{t^{n-1}}{n!}\tilde{X}^n \mathsf{id}(z)\| \\ &\le |t|\sum_2^\infty \frac{n^n}{n!}|t|^{n-2}(\frac{\|X\|}{r})^n \le |t|M \to 0,\end{aligned}$$

where M does not depend on $z \in C$ or t. We refer to [11] for the definition and properties of the *Carathéodory distance* as well as for the background material on JB*-triples.

2. Holomorphic automorphisms and equicontinuous sets

Let $X \in \mathsf{aut}(B_F)$ and $\mathcal{E} \subset F$ be given. Then $\tilde{X}\mathsf{id}\colon F \to F$ is an entire function, therefore

$$\tilde{X}\mathcal{E} := \{\tilde{X}\mathsf{id}(f) \mid f \in \mathcal{E}\} = \{X(f) \mid f \in \mathcal{E}\} \subset F,$$

and one can speak of the equicontinuity of $\tilde{X}\mathcal{E}$. Similar considerations can be made on

$$\tilde{X}^n\mathcal{E} := \{\tilde{X}^n\mathsf{id}(f) \mid f \in \mathcal{E}\}, \qquad n > 1.$$

2.1 Definition. i) An element $\phi \in \mathsf{Aut}(B_F)$ is called an *equiautomorphism* of B_F if it preserves the family of all subsets $\mathcal{E} \subset B_F$ that are equicontinuous and bounded away from the boundary ∂B_F. ii) An element $X \in \mathsf{aut}(B_F)$ is called an *equivector field* if it preserves the family of all bounded equicontinuous sets $\mathcal{E} \subset F$.

The set $\mathsf{Eaut}(B_F)$ of all equiautomorphisms of B_F is clearly a closed subgroup of $\mathsf{Aut}(B_F)$ and $\mathsf{eaut}(B_F)$, the set of all complete equivector fields, is a closed Lie-subalgebra of $\mathsf{aut}(F)$.

2.2 Proposition. *A continuous oneparameter group $t \mapsto \phi_t \in \mathsf{Aut}(B_F)$ satisfies $\phi_t \in \mathsf{Eaut}(B_F)$ for all $t \in \mathbb{R}$ if and only if its infinitesimal generator X belongs to* $\mathsf{eaut}(F)$.

Proof. Assume that $\phi_t \in \mathsf{Eaut}(B_F)$ for all $t \in \mathbb{R}$ and let X be its infinitesimal generator. Let $\mathcal{E} \subset F$ be any bounded equicontinuous set. Replacing $\mathcal{E}$ by a suitable scalar multiple one can assume that $\mathcal{E} \subset B_F$ and that $r := \mathrm{dist}(\mathcal{E}, \partial B_F) > 0$. We show that $\tilde{X}\mathcal{E}$ is (obviously bounded) equicontinuous at every point $x_0 \in \Omega$. Let $\varepsilon > 0$ be given. For $f \in \mathcal{E}$, $y \in \Omega$ and $t \in \mathbb{R}$ one has

$$\begin{aligned} Xf(y) - Xf(x_0) = [Xf(y) - \frac{1}{t}\big(\phi_t f(y) - f(y)\big)] \\ +[\frac{1}{t}\big(\phi_t f(y) - f(y)\big) - \frac{1}{t}\big(\phi_t f(x_0) - f(x_0)\big)] \\ + [\frac{1}{t}\big(\phi_t f(x_0) - f(x_0)\big) - Xf(x_0)]. \end{aligned}$$

By (1.2) there is a value of t (say T) which does not depend on $f \in \mathcal{E}$ or $y \in \Omega$ such that

$$\|Xf(y) - \frac{1}{T}\big(\phi_T f(y) - f(y)\big)\| \le \varepsilon.$$

By assumption the family $\{\phi_T f \mid f \in \mathcal{E}\}$ is equicontinuous, hence so is the family $\{\frac{1}{T}\big(\phi_T f - f\big) \mid f \in \mathcal{E}\}$. Therefore there is a neighbourhood V_{x_0} of x_0 in Ω such that whenever $f \in \mathcal{E}$ and $y \in V_{x_0}$

$$\|\frac{1}{T}\big(\phi_T f(y) - f(y)\big) - \frac{1}{T}\big(\phi_T f(x_0) - f(x_0)\big)\| \le \varepsilon.$$

Thus $\|Xf(y) - Xf(x_0)\| \le 3\varepsilon$ which shows the equicontinuity of $\tilde{X}\mathcal{E}$ at x_0.

Conversely, let $X \in \mathsf{eaut}(B_F)$ be given and let $\mathcal{E} \subset B_F$ be an equicontinuous set that is bounded away from ∂B_F. Set $r := \mathrm{dist}(\mathcal{E}, \partial B_F) > 0$. Let $\varepsilon > 0$ be given and fix any $x_0 \in \Omega$. By (1.1) for $|t| \cdot \|X\| < r$ there is an index n_0 not depending on $f \in \mathcal{E}$ such that

$$\sum_{n_0}^{\infty} \frac{k^k}{k!}\big(\frac{|t| \cdot \|X\|}{r}\big)^k \le \varepsilon.$$

By assumption $\mathcal{E}$, $\tilde{X}\mathcal{E}, \ldots, \tilde{X}^{n_0}\mathcal{E}$ are equicontinuous sets, hence there is a neighbourhood V_{x_0} of x_0 in Ω such that

$$\|\frac{1}{k!}\tilde{X}^k \mathsf{id} f(y) - \frac{1}{k!}\tilde{X}^k \mathsf{id} f(x_0)\| \le \frac{\varepsilon}{n_0 + 1}$$

for all $0 \le k \le n_0$, all $f \in \mathcal{E}$ and all $y \in V_{x_0}$. Therefore

$$\begin{aligned} \|\phi_t f(y) - \phi_t f(x_0)\| \le \sum_{0}^{n_0} \|\frac{1}{k!}\tilde{X}^k \mathsf{id} f(y) - \frac{1}{k!}\tilde{X}^k \mathsf{id} f(x_0)\| \\ + 2\sum_{n_0+1}^{\infty} \frac{k^k}{k!}\big(\frac{|t|.\|X\|}{r}\big)^k \le 3\varepsilon \end{aligned}$$

holds for all $f \in \mathcal{E}$ and all $y \in V_{x_0}$. This shows that for small values of t (hence for all $t \in \mathbb{R}$), $\phi_t \mathcal{E}$ is equicontinuous at x_0. Since $\mathrm{dist}(\mathcal{E}, \partial B_F) > 0$ and the Carathéodory distance on B_F is invariant under the group $\mathsf{Aut}(B_F)$ and it is equivalent to the norm distance on every subset of the ball that is bounded away from the boundary, one can show that $\mathrm{dist}(\phi_t \mathcal{E}, \partial B_F) > 0$. □

We have the following commutative diagram where the horizontal arrows are the canonical inclusions:

$$\begin{array}{ccc} \mathsf{Eaut}^0(B_F) & \xrightarrow{Id} & \mathsf{Aut}^0(B_F) \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathsf{eaut}(B_F) & \xrightarrow{id} & \mathsf{aut}(B_F). \end{array}$$

2.3 Corollary. $\mathsf{Eaut}^0(B_F)$ *is a real Banach-Lie group whose Banach-Lie algebra is* $\mathsf{eaut}(B_F)$.

Proof. It follows easily from (2.2). □

2.4 Lemma. *Let the JB*-triples E and F be as above. Then* i) *Every transvection in* $\mathsf{aut}(B_F)$ *is a complete equivector field.* ii) *Every Moebius transformation of B_F is an equiautomorphism.*

Proof. By (2.2) it suffices to prove i). Let $\mathcal{E} \subset F$ and $p_a \in \mathfrak{P}$ be a bounded equicontinuous set and a transvection respectively. Then $p_a\mathcal{E} := \{a - \{faf\} \mid f \in \mathcal{E}\}$ is equicontinuous. □

Unlike transvections, there are derivations of F that do not preserve the family of all equicontinuous subsets of F (see example (3.4) in §3). Thus we define

2.5 Definition. i) An element $\phi \in \mathsf{Isom}(F)$ is called an *equiisometry* if for every equicontinuous set $\mathcal{E} \subset F$ the image $\phi\mathcal{E}$ is equicontinuous. ii) An element $\delta \in \mathsf{isom}(F)$ is called an *equiderivation* of F if for every bounded equicontinuous set $\mathcal{E} \subset F$ the image $\delta\mathcal{E}$ is equicontinuous.

The set $\mathsf{Eisom}(F)$ of all equiisometries of of F is a closed subgroup of $\mathsf{Isom}(F)$ and the set $\mathsf{eisom}(F)$ of all equiderivations of F is a closed real Lie subalgebra of $\mathsf{isom}F$.

2.6 Proposition. *A one parameter group $t \mapsto \phi_t \in \mathsf{Isom}(F)$ satisfies $\phi_t \in \mathsf{Eisom}(F)$ for all $t \in \mathbb{R}$ if and only if its infinitesimal generator δ satisfies $\delta \in \mathsf{eisom}(F)$.*

Proof. Repeat the argument in (2.2) with X replaced by δ. □

2.7 Corollary. $\mathsf{Eisom}^0(F)$ *is a real Banach-Lie group whose Banach-Lie algebra is* $\mathsf{eisom}(F)$.

Proof. It follows easily from (2.6). □

We do not know whether $\mathsf{Eisom}(F)$ is an algebraic subgroup of $\mathsf{Isom}(F)$. More generally, by [11], prop.8.13, $\mathsf{Eisom}(F)$ is a Lie subgroup of $\mathsf{Isom}(F)$ if and only if $\mathsf{eisom}(F)$ is a split subspace of $\mathsf{isom}(F)$ and it would be interesting to find conditions for this to happen.

2.8 Corollary. *The following topologically direct vector space decomposition holds*

$$\mathsf{eaut}(B_F) = \mathsf{eisom}(F) \oplus \mathfrak{p},$$

where $\mathfrak{p}$ is the set of all transvections of B_F. Similarly, for $\mathfrak{P} := \exp \mathfrak{p}$ we have

$$\mathsf{Eaut}(B_F) = \mathsf{Eisom}(F) \cdot \mathfrak{P} = \mathfrak{P} \cdot \mathsf{Eisom}(F).$$

Proof. It follows easily from (1.0) and (2.6). □

We establish the following isomorphisms of Banach-Lie algebras and Banach-Lie groups. See [10], prop. 2.1, for the case $\dim E < \infty$:

$$\mathsf{eisom}(F) \sim \mathcal{C}(\Omega,\, \mathsf{isom}(E)) \qquad \mathsf{Eisom}^0(F) \sim \mathcal{C}(\Omega,\, \mathsf{Isom}^0(E))$$

$$\mathsf{eaut}(B_F) \sim \mathcal{C}(\Omega,\, \mathsf{aut}(B_E)) \qquad \mathsf{Eaut}^0(B_F) \sim \mathcal{C}(\Omega,\, \mathsf{Aut}^0(B_E))$$

Recall ([1], Remark 4.4) that if Z is a JB*-triple, $\delta \in \mathsf{isom}(Z)$ and $J \subset Z$ is an ideal, then $\delta J \subset J$. Now fix $x \in \Omega$ and $\delta \in \mathsf{isom}(F)$. We define $\delta^x \colon E \to E$ by $\delta^x := \varepsilon_x \circ \delta \circ \varepsilon_x^{-1}$, where $\varepsilon_x \colon \mathcal{C}(\Omega, E) \to E$, the evaluation at x, is a triple homomorphism. Hence $J := \ker(\varepsilon_x)$ is a triple ideal and so $\delta(\ker(\varepsilon_x)) \subset \ker(\varepsilon_x)$ which shows that δ^x is a well defined map. It is a routine to show that $\delta^x \in \mathsf{isom}(E)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(\Omega,\, E) & \xrightarrow{\varepsilon_x} & E \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\delta^x} \\ \mathcal{C}(\Omega,\, E) & \xrightarrow{\varepsilon_x} & E. \end{array}$$

Let us define $\hat{\delta} \colon \Omega \to \mathsf{isom}(E)$ by $\hat{\delta}(x) := \delta^x,\ x \in \Omega$.

Remark. Every vector $z \in E$ can be considered as an element of $\mathcal{C}(\Omega, E)$ by identifying z with the constant function $f(x) = z$ and whenever this is done we stress it by writing $\mathbf{z}$ in boldface characters. We shall frequently use that $\delta^x(z) = \delta\mathbf{z}(x)$.

2.9 Proposition. *The mapping $\lambda \colon \delta \mapsto \hat{\delta}$ is a surjective isometric isomorphism of the Banach-Lie algebras $\mathsf{eisom}(F)$ and $\mathcal{C}(\Omega,\, \mathsf{isom}(E))$.*

Proof. We divide the proof into several steps.
Step 1. For every fixed δ, $\hat{\delta} \colon \Omega \to \mathsf{isom}(E)$ is continuous. Fix any point $x_0 \in \Omega$. Let $\varepsilon > 0$ be given. Identifying each $w \in E$ such that $\|w\| \le 1$ with the constant function $\mathbf{w}$ one has

$$\|\delta^y w - \delta^{x_0} w\| = \|(\delta\mathbf{w})y - (\delta\mathbf{w})x_0\|.$$

The set $\mathcal{E} := \{\mathbf{w} \mid w \in E, \|w\| \le 1\}$ is clearly bounded and equicontinuous, hence so is $\{\delta\mathbf{w} :\ \|w\| \le 1\}$. Therefore there is a neighbourhood V_{x_0} of x_0 in Ω such that

$$\|\delta^y w - \delta^{x_0} w\| = \|(\delta\mathbf{w})y - (\delta\mathbf{w})x_0\| \le \varepsilon$$

holds for all $w \in E$ with $\|w\| \le 1$ and all $y \in V_{x_0}$. Then, for all $y \in V_{x_0}$ we have

$$\|\delta^y - \delta^{x_0}\| = \sup_{\|w\| \le 1} \|\delta^y w - \delta^{x_0} w\| \le \varepsilon.$$

Step 2. $\lambda \colon \delta \to \hat{\delta}$ is a continuous Lie algebra homomorphism

$$\mathsf{eisom}(F) \to \mathcal{C}(\Omega, \mathsf{isom}(E)).$$

It is easy to show that λ is a Lie algebra homomorphism and $\|\lambda\| \le 1$ since

$$\begin{aligned} \|\lambda(\delta)\| &= \sup_{x \in \Omega} \|\delta^x\| = \sup_{x \in \Omega} \sup_{\|z\| \le 1} \|\delta^x(z)\| \\ &= \sup_{x \in \Omega} \sup_{\|\mathbf{z}\| \le 1} \|(\delta\mathbf{z})x\| \le \sup_{\|\mathbf{z}\| \le 1} \sup_{x \in \Omega} \|\delta\|\, \|\mathbf{z}(x)\| \le \|\delta\|. \end{aligned}$$

Step 3. λ is surjective. Let $\psi \in \mathcal{C}(\Omega,\, \mathsf{isom}(E))$ be given. For each fixed $f \in \mathcal{C}(\Omega, E)$ we define another continuous function $\delta_\psi f \colon \Omega \to E$ by $\delta_\psi f(x) := \psi(x) f(x),\ x \in \Omega$. It is a routine to show that δ_ψ is a derivation of $F = \mathcal{C}(\Omega,\, E)$, that δ_ψ belongs to $\mathsf{eisom}(F)$ and $\lambda(\delta_\psi) = \psi$.

Step 4. Clearly the mapping $\psi \to \delta_\psi$ is linear and λ is an isometry since

$$\begin{aligned}\|\delta_\psi\| = \sup_{\|f\|\le 1} \|\delta_\psi f\| &= \sup_{x\in\Omega} \sup_{\|f\|\le 1} \|\delta_\psi f(x)\| \\ &= \sup_{x\in\Omega} \sup_{\|f\|\le 1} \|\psi(x)\delta_\psi f(x)\| \le \sup_{x\in\Omega} \|\psi(x)\| = \|\psi\|.\end{aligned}$$

This completes the proof. □

It is known ([9], p. 30) that for every Banach-Lie group G with Banach-Lie algebra $\mathfrak{g}$, the space $\mathcal{C}(\Omega,\, \mathsf{G})$ with the pointwise operations, the compact-open topology and a suitable manifold structure is a Banach-Lie group whose Lie algebra is $\mathcal{C}(\Omega,\, \mathfrak{g})$. Applying this to $\mathsf{G} := \mathsf{Isom}^0(E)$ and $\mathfrak{g} := \mathsf{isom}(E)$ we get the existence of a unique Banach-Lie group isomorphism that makes the following diagram commutative:

$$\begin{array}{ccc} \mathsf{Eisom}^0(F) & \xrightarrow{\Lambda} & \mathcal{C}(\Omega, \mathsf{Isom}^0(E)) \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathsf{eisom}(F) & \xrightarrow{\lambda} & \mathcal{C}(\Omega, \mathsf{isom}(E)). \end{array}$$

2.10 Theorem. *With the precedent notation,* $\mathsf{Eaut}^0(B_F)$ *and* $\mathcal{C}(\Omega, \mathsf{Aut}^0(B_E))$ *are isomorphic as real Banach-Lie groups.*

Proof. Using the decomposition in (2.8), λ extends to an isomorphism between $\mathsf{eaut}(B_F)$ and $\mathcal{C}(\Omega,\, \mathsf{aut}(B_E))$. For that purpose define $\big(\lambda(a-a^*)\big)(x) := a(x) - a(x)^*,\ a \in F,\ x\in\Omega$. Thus there exists a unique Banach-Lie group isomorphism that makes the following diagram commutative:

$$\begin{array}{ccc} \mathsf{Eaut}^0(B_F) & \xrightarrow{\Lambda} & \mathcal{C}(\Omega, \mathsf{Aut}^0(B_E)) \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathsf{eaut}(B_F) & \xrightarrow{\lambda} & \mathcal{C}(\Omega, \mathsf{aut}(B_E)). \end{array}$$

This completes the proof. □

An explicit expression for Λ in $\mathfrak{p}_F$ can be obtained in this way: Let $\phi\colon \mathbb{R} \to \mathcal{C}(\Omega,\, E)$ be a Fréchet differentiable function; then $\varepsilon_x \frac{d}{dt}\phi_t = \frac{d}{dt}\varepsilon_x\phi_t$. Given $a\in F$, we recall that $\phi_t f := \exp t(a-a^*)f$ is the maximal solution in F of the initial value problem $\frac{d}{dt}\phi_t = a - \{\phi_t a \phi_t\},\ \phi_0 = f$. Applying here ε_x one gets that $\varphi_t := \varepsilon_x\phi_t = \phi_t(x)$ satisfies in E the differential equation $\frac{d}{dt}\varphi_t = a(x) - \{\varphi_t a(x)\varphi_t\},\quad \varphi_0(x) = f(x)$. In other words, one has

$$\big(\exp t(a-a^*)f\big)(x) = \exp t\big(a(x)-a(x)^*\big)f(x). \tag{2.10}$$

For a JB*-triple Z and $\alpha\in Z$, $z\in B$, it is customary to write $\beta := \exp(\alpha-\alpha^*)0$, $M_\beta(z) := \exp(\alpha-\alpha^*)z$. With this notation (2.10) is written for $t=1$

$$b(x) := \exp\big(a(x)-a(x)^*\big)0, \qquad (M_b f)(x) = M_{b(x)}f(x), \qquad x\in\Omega, \qquad f\in B_F,$$

which gives the formula $\Lambda\colon \mathfrak{P}_F \leftrightarrow \mathcal{C}(\Omega,\, \mathfrak{P}_E)$ we were looking for.

3. Holomorphic automorphisms and the Banach-Stone property of E

We recall ([3], p. 142) that a complex Banach space E is said to have the *strong Banach-Stone property* (the sBSp for short) if for every pair M, N of locally compact Hausdorff topological spaces and for every surjective linear isometry $\phi\colon \mathcal{C}_0(M, E) \to \mathcal{C}_0(N, E)$ there are a homeomorphism $\tau\colon N \to M$ and a continuous function $u\colon N \to \mathsf{Isom}(E)$ such that

$$(\phi f)(x) = u(x)\, f[\tau(x)], \qquad f \in \mathcal{C}_0(M, E), \qquad x \in N. \tag{3.0}$$

Notice that here $\mathsf{Isom}(E) \subset \mathsf{GL}(E)$ is endowed with the *strong operator topology* which in general is weaker than the norm topology.

3.1 Example. 1. Every complex Banach space E whose *centralizer* $Z(E)$ is one-dimensional has the sBSp ([3], th. 8.11). Every Cartan factor E satisfies the above condition since by [5], th. 2.8, for JB*-triples the centralizer is the same as the centroid which for irreducible JBW*-triples is one-dimensional ([5], cor. 2.11).

2. Every complex Banach space E with no nontrivial M-ideals has the sBSp ([3], prop. 5.1). For a JB*-triple E, M-ideals are the same as closed triple ideals ([2], th. 3.2), hence every simple J*-algebra ([6], p. 347) has the sBSp. In particular that is case for the space $E := \mathcal{K}(H)$ of all compact operators on a Hilbert space H.

3.2 Theorem. *Let E be a JB*-triple with the sBSp and set $F := \mathcal{C}(\Omega, E)$ where Ω is a compact Hausdorff space. Then the following statements are equivalent:*

i) *For every $\phi \in \mathsf{Isom}^0(F)$, the function $u\colon \Omega \to \mathsf{Isom}(E)$ given by (3.0) is continuous when $\mathsf{Isom}(E)$ is endowed with the norm topology.*

ii) *Every isometry $\phi \in \mathsf{Isom}^0(F)$ maps every bounded equicontinuous subset $\mathcal{E} \subset F$ into an equicontinuous set.*

iii) *We have $\mathsf{Isom}^0(F) \sim \mathcal{C}(\Omega, \mathsf{Isom}^0(E))$ as Banach-Lie groups.*

iv) *We have $\mathsf{Aut}^0(B_F) \sim \mathcal{C}(\Omega, \mathsf{Aut}^0(B_E))$ as Banach-Lie groups.*

Proof. i) $\Rightarrow$ ii). Let $\phi \in \mathsf{Isom}^0(F)$, whence by (3.0) one has $\tau = \mathsf{id}_\Omega$. Let $\mathcal{E} \subset F$ be bounded and equicontinuous, and fix $x_0 \in \Omega$ and $\varepsilon > 0$. By the norm continuity of u and the equicontinuity of $\mathcal{E}$, there is neigbourhood V_{x_0} of x_0 in Ω such that

$$\|u(y) - u(x_0)\| \le \varepsilon, \qquad \|f(y) - f(x_0)\| \le \varepsilon$$

hold for all $y \in V_{x_0}$ and $f \in \mathcal{E}$. If M is an upper bound of $\mathcal{E}$, then the identity

$$\begin{aligned}\phi f(y) - \phi f(x_0) &= u(y)f(y) - u(x_0)f(x_0)\\ &= u(y)\big(f(y) - f(x_0)\big) + \big(u(y) - u(x_0)\big)f(x_0)\end{aligned}$$

yields $\|\phi f(y) - \phi f(x_0)\| \le (1 + M)\varepsilon$ for $f \in \mathcal{E}$ and $y \in V_{x_0}$ hence ii) holds.

ii) $\Rightarrow$ iii). The assumption means $\mathsf{Isom}^0(F) \subset \mathsf{Eisom}^0(F)$ hence equality holds and iii) follows from the discussion in the proof of (2.9).

iii) $\Rightarrow$ iv). One has

$$\begin{aligned}\mathsf{Aut}^0(B_F) &= \mathsf{Isom}^0(F) \cdot \mathfrak{P}_F && \text{by (1.0) applied to } Z := F\\ \mathsf{Isom}^0(F) &= \mathsf{Eisom}^0(F) && \text{by assumption}\\ \mathsf{Eisom}^0(F) &\sim \mathcal{C}(\Omega, \mathsf{Isom}^0(E)) && \text{by (2.9)}\\ \mathfrak{P}_F &\sim \mathcal{C}(\Omega, \mathfrak{P}_E) && \text{as seen in (2.10).}\end{aligned}$$

Thus combining everything

$$\begin{aligned}\mathsf{Aut}^0(B_F) = \mathsf{Isom}^0(F)\cdot\mathfrak{P}_F &\sim \mathcal{C}(\Omega,\, \mathsf{Isom}^0(E))\cdot\mathcal{C}(\Omega,\, \mathfrak{P}_E)\\ &\sim \mathcal{C}(\Omega,\, \mathsf{Isom}^0(E)\cdot\mathfrak{P}_E) \sim \mathcal{C}(\Omega,\, \mathsf{Aut}^0(B_E)).\end{aligned}$$

iv) $\Rightarrow$ *i)*. If in $\mathsf{Aut}^0(B_F) \sim \mathcal{C}(\Omega,\, \mathsf{Aut}^0(B_E))$ we take the identity component of the isotropy subgroups at the origin we get $\mathsf{Isom}^0(F) \sim \mathcal{C}(\Omega,\, \mathsf{Isom}^0(E))$. Hence those isometries of F which are close to id_F have the form $u\colon \Omega \to \mathsf{Isom}^0(E)$ for some continuous function u. Here $\mathsf{Isom}^0(E)$ has the topology inherited from $\mathsf{Aut}^0(B_E)$, that is the uniform topology. Thus i) holds. □

3.3 Corollary. *If* $\dim E < \infty$*, then* $\mathsf{Aut}^0(B_F) \sim \mathcal{C}(\Omega,\, \mathsf{Aut}^0(B_E))$ *as Banach-Lie groups.*

Proof. In that case E has the sBSp and the strong operator topology on $\mathsf{GL}(E)$ is the same as the uniform topology; so the result follows from (3.2). □

3.4 Example. Suppose we had a JB*-triple E with the sBSp, a compact space Ω and a function $u\colon \Omega \to \mathsf{Isom}(E)$ which is continuous when $\mathsf{Isom}(E)$ carries the strong operator topology but it is discontinuous if $\mathsf{Isom}(E)$ carries the uniform topology. By the sBSp of E we would have $u \in \mathsf{Isom}^0(F)$ whereas by (3.2) $u \notin \mathsf{Eisom}^0(F)$. Now we construct such a function. Let $\Omega := \{e^{it} \mid t \in \mathbb{R}\}$ be the unit circle of the complex plane and $E := L^2(\Omega)$. Then E is a separable Hilbert space, hence it has the sBSp. The set $\mathcal{T}$ of trigonometric polynomials

$$\sqrt{2\pi}e_n(t) := e^{int}, \qquad \pm n \in \mathbb{N}, \qquad t \in \Omega,$$

is an orthonormal basis in E. For every $x \in \mathbb{R}$ we define $e_n(\cdot, x)$ as the result of *rotating* $e_n(\cdot)$ by x, that is $e_n(t, x) := e^{in(t+x)}$. Since the Lebesgue measure is rotation invariant, the $e_n(\cdot, x)$ form an orthonormal basis in E and one can define $u(x) \in \mathsf{Isom}(E)$ as the unitary operator given by $u(x)e_n(\cdot) := e_n(\cdot, x)$. Then we have a function $u\colon \Omega \to \mathsf{Isom}(E)$ that meets the requirements:

Continuity in the strong operator topology: Let $h \in E$ be given and let $h = \sum h_n e_n(\cdot)$ be its Fourier series with respect to the basis $\mathcal{T}$. Then

$$\begin{aligned}\big(u(y) - u(x_0)\big)h &= \sum \big(u(y) - u(x_0)\big)h_n e_n(\cdot)\\ &= \sum \big(e^{iny} - e^{inx_0}\big)h_n e_n(\cdot) = \sum \alpha_n e_n(\cdot),\end{aligned}$$

where the α_n are defined in an obvious way. Let $\varepsilon > 0$ be given. Since $\sum|\alpha_n|^2$ is summable, there is an index n_0 such that $\sum_{|n|>n_0}|\alpha_n|^2 < \varepsilon$ and by the continuity of $y \mapsto e^{iny}$ we may take y in a small neigbourhood of x_0 in Ω so that

$$\|\big(u(y) - u(x_0)\big)h\|^2 = \sum_{|n|<n_0} |h_n|^2\cdot|e^{iny} - e^{inx_0}|^2 + \sum_{|n|\ge n_0+1} |\alpha_n|^2 < 2\varepsilon.$$

Discontinuity in the uniform topology: One has

$$2\pi\|\big(u(y) - u(x_0)\big)e_n(\cdot)\|^2 = \int_0^{2\pi} |e^{in(t+y)} - e^{in(t+x_0)}|^2 dt = 2\pi\,|1 - e^{in(y-x_0)}|^2.$$

Let $\varepsilon > 0$ be small. Then

$$\|u(y) - u(x_0)\|^2 = \sup_{\|h\| \leq 1} \|\big(u(y) - u(x_0)\big)h\| \geq \sup_n \|\big(u(y) - u(x_0)\big)e_n(\cdot)\|$$
$$= \sup_n |1 - e^{in(y-x_0)}|^2 > \varepsilon$$

unless $y = x_0$ as one easily sees.

References

1. Barton, T.J., Friedman, Y.: Bounded derivations of JB*-triples. Quart. J. Math. Oxford **41**, 255-268 (1990). MR **91j:**46086
2. Barton, T.J., Timoney, R. M.: Weak*-continuity of Jordan triple products and applications, Math. Scand. **59**, 177-191 (1986). MR **88d:**46129
3. Behrends, E.: M-Structure and the Banach-Stone theorem, Lecture Notes in Mathematics Vol. 736. Berlin-Heidelberg-New York: Springer Berlin-Heidelberg-New York: Springer 1979. MR **81b:**46002
4. Braun, R. , Kaup, W. , Upmeier, H.: On the automorphisms of circular and Reinhardt domains in complex Banach spaces, Manuscripta Math. **25**, 97-133 (1978). MR **80g:**32003
5. Dineen, S., Timoney, R.M.: The centroid of a JB*-triple system. Math. Scand. **62**, 327-342 (1988). MR **89j:**46067
6. Harris, L.A.: A generalization of C*-algebras. Proc. London Math. Soc. **42**, 331-361 (1981). MR **82e:**46089
7. Harris, L.A., Kaup, W.: Linear algebraic groups in infinite dimensions. Illinois J. Math. **21**, 666-674 (1977). MR **57:**544
8. Kaup, W.: A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. Math. Z. **183**, 503-529 (1983). MR **85c:**46040
9. Mellon, P.: Dual manifolds of JB*-triples of the form $\mathcal{C}(X, U)$. Proc. R. Ir. Acad. **93 A**, 27-42 (1993). MR **94k:**58011
10. Mellon, P.: Symmetric manifolds of compact type associated to the JB*-triples $\mathcal{C}_0(X, Z)$, Math. Scand. **78**, 19-36 (1996). MR **97h:**46112
11. Upmeier, H.: Symmetric Banach manifolds and Jordan C*-algebras, North Holland Math. Studies **104** North-Holland, Amsterdam 1985. MR **87a:**58022

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago, Santiago de Compostela, Spain

E-mail address: jmisidro@zmat.usc.es