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A CLASS OF $P^t(d\mu)$ SPACES WHOSE POINT EVALUATIONS VARY WITH t

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Abstract. Extending an example given by T. Kriete, we develop a class of measures each of which consists of a measure on $\{z : |z| = 1\}$ along with a series of weighted point masses in $\mathbf{D} := \{z : |z| < 1\}$. This class provides relatively simple examples of measures μ which have the property that the collection of analytic bounded point evaluations for $P^t(d\mu)$ varies with t. The first known measures with this property were recently constructed by J. Thomson.

1. Introduction

As an application of some of the general theory concerning reproducing kernel Hilbert spaces developed in [3], T. Kriete produced a finite, positive Borel measure μ (see [2], Special case (i), p.193) that consists of a measure wdm (on $\{z : |z| = 1\}$) – absolutely continuous with respect to m (normalized Lebesgue measure on $\{z : |z| = 1\}$) – along with a sum of weighted point masses $\sigma := \sum_{n=1}^{\infty} c_n \delta_{z_n} (0 < z_1 < z_2 < \cdots < z_n \to 1$ as $n \to \infty)$ such that :

1) $\int \log(w) dm = -\infty$, and so by Szegö's Theorem the polynomials are dense in $L^2(wdm)$; however

2) $\frac{1}{z} \notin P^2(d\mu)$ – the closure of the polynomials in $L^2(d\mu)$.

In essence, the series of weighted point masses σ plugs the weakness in wdm with the result that every function in $P^2(d\mu)$ has an analytic continuation to $\mathbf{D} := \{z : |z| < 1\}$ – that is, the set of analytic bounded point evaluations for $P^2(d\mu)$ (denoted by $abpe(P^2(d\mu))$) equals $\mathbf{D}$. In this paper we use rather elementary function theoretic methods to construct a variety of measures μ of the type constructed by T. Kriete and we examine $abpe(P^t(d\mu))$ for $1 \le t < \infty$. Among these measures μ are relatively simple examples for which $abpe(P^t(d\mu))$ varies with t. Indeed, for any $\lambda, 1 < \lambda < \infty$, we give (see Theorem 3.2) a measure μ of the type produced by T. Kriete such that $abpe(P^t(d\mu)) = \emptyset$ if $1 \le t < \lambda$ and $abpe(P^t(d\mu)) = \mathbf{D}$ if $t \ge \lambda$. Until now, the only known examples of $P^t(d\mu)$ spaces for which $abpe(P^t(d\mu))$ varies with t are those that were recently constructed by J. Thomson (see [5]) who made use of K. Seip's work on sampling and interpolation in Bergman spaces. Later in this paper we examine the general question concerning which measures wdm have weakness

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that can be "plugged" by a series of weighted point masses that lie on some radial segment in $\mathbf{D}$ and we also discuss nonradial analogues.

2. Preliminaries

For any finite, positive Borel measure μ with compact support in the complex plane $\mathbf{C}$ and for $1 \leq t < \infty$, let $P^t(d\mu)$ denote the closure of the polynomials in $L^t(d\mu)$. A point z in $\mathbf{C}$ is called a *bounded point evaluation* for $P^t(d\mu)$ if there is a constant c such that $|p(z)| \leq c||p||_{L^t(d\mu)}$ for all polynomials p; let $bpe(P^t(d\mu))$ denote the collection of all such points. If $z \in bpe(P^t(d\mu))$, then by the Hahn-Banach and Riesz Representation Theorems there exists k_z in $L^s(d\mu)$ $(\frac{1}{s} + \frac{1}{t} = 1)$ such that $p(z) = \int p(\xi)k_z(\xi)d\mu(\xi)$ for all polynomials p. If $f \in P^t(d\mu)$, then define $\hat{f}$ on $bpe(P^t(d\mu))$ by $\hat{f}(z) = \int f(\xi)k_z(\xi)d\mu(\xi)$; observe that $\hat{f} = f$ *a.e.* μ on $bpe(P^t(d\mu))$. A point z in $\mathbf{C}$ is called an *analytic bounded point evaluation* for $P^t(d\mu)$ if there are positive constants M and r such that $|p(w)| \leq M||p||_{L^t(d\mu)}$ for all polynomials p and all w such that $|z-w| < r$; the set of all points z of this type is denoted by $abpe(P^t(d\mu))$. Now $abpe(P^t(d\mu))$ is an open subset of $bpe(P^t(d\mu))$ and, by the Maximum Modulus Theorem, each component of $abpe(P^t(d\mu))$ is simply connected. Observe that $z \longmapsto \hat{f}(z)$ is analytic on $abpe(P^t(d\mu))$ for each f in $P^t(d\mu)$. J. Thomson has given a direct sum decomposition of $P^t(d\mu)$ that involves the components of $abpe(P^t(d\mu))$ (see [4], Theorem 5.8). We end this section with a rather well-known result concerning analytic bounded point evaluations; we include it and its proof for the sake of completeness.

Lemma 2.1. *Let μ be a finite, positive Borel measure with compact support in $\mathbf{C}$. Suppose $z \notin \text{support}(\mu)$ and let U be the component of $\mathbf{C} \setminus \text{support}(\mu)$ such that $z \in U$. If $z \in bpe(P^t(d\mu))$, then $U \subseteq abpe(P^t(d\mu))$.*

Proof. Now since $z \in bpe(P^t(d\mu))$, it follows that $\zeta \longmapsto \frac{1}{z-\zeta} \notin P^t(d\mu)$. Applying the Hahn-Banach Theorem, there exists g in $L^s(d\mu)$ $(\frac{1}{s}+\frac{1}{t} = 1)$ such that $\int pg d\mu = 0$ for all polynomials p and yet $\int \frac{g(\zeta)}{z-\zeta}d\mu(\zeta) \neq 0$. Therefore, the Cauchy transform $\hat{g}(w) := \int \frac{g(\zeta)}{\zeta-w}d\mu(\zeta)$ is analytic and not identically zero in U. Choose z' in U and select $r > 0$ such that $\{w : |z' - w| \leq r\} \subseteq U$ and $|\hat{g}(w)| \geq \epsilon > 0$ whenever $|z' - w| = r$. Since $g \perp P^t(d\mu)$, if $w \in U$ and p is any polynomial, then

$$p(w)\hat{g}(w) := \int \frac{p(\zeta)g(\zeta)}{\zeta - w}d\mu(\zeta).$$

So, by Hölder's Inequality and our choice of r, there is a constant M such that $|p(w)| \leq M||p||_{L^t(d\mu)}$ whenever $|z' - w| = r$ and p is a polynomial. By the Maximum Modulus Theorem we conclude that $z' \in abpe(P^t(d\mu))$ and hence $U \subseteq abpe(P^t(d\mu))$. □

3. A class of $P^t(d\mu)$ spaces

We now introduce a collection of measures of the type produced by T. Kriete. Let φ be the Möbius transformation from $\mathbf{D} := \{z : |z| < 1\}$ onto $\{\zeta : Re(\zeta) > 0\}$ given by $\varphi(z) = \frac{1+z}{1-z}$ and for any constant $c \geq 1$ define g_c on $\mathbf{D}$ by $g_c(z) = \cos(\frac{\pi}{3c}\varphi(z))$. Notice that g_c extends continuously to $\overline{D}\setminus\{1\}$. Let ν_c be the measure on $\partial\mathbf{D}$ given by $d\nu_c = \frac{1}{|g_c|}dm$, where m is normalized Lebesgue measure on $\partial\mathbf{D}$, and let σ_c be the sum of weighted point masses $\sum_{n=1}^{\infty}\alpha_n\delta_{z_n}$, where $z_n = \varphi^{-1}(3c(n-\frac{1}{2}))$ ($\{z_n\}$ are the zeros of g_c in $\mathbf{D}$) and $\alpha_n = \frac{1}{|z_n||g_c'(z_n)|}$. Observe that:

$$e^{\frac{-\pi}{3c}(\frac{|\sin\theta|}{1-\cos\theta})} \le \frac{1}{|g_c(e^{i\theta})|} \le 2e^{\frac{-\pi}{3c}(\frac{|\sin\theta|}{1-\cos\theta})} \text{ for } 0<\theta<2\pi; \tag{3.1}$$

$$0<z_1<z_2<\cdots<z_n\to 1 \text{ as } n\to\infty \text{ and } \frac{1}{|g_c'(z_n)|} = \frac{6c}{\pi(3c(n-\frac{1}{2})+1)^2}. \tag{3.2}$$

From (3.1) and (3.2) we see that both ν_c and σ_c are finite, positive Borel measures. Also from (3.1) it is clear that $\int\log(\frac{1}{|g_c|})dm = -\infty$ and hence, by Szegö's Theorem, $P^t(d\nu_c) = L^t(d\nu_c)$ for $1\le t<\infty$. For constants c and d both greater than or equal to 1, let $\mu_{c,d} = \nu_c + \sigma_d$. Our first result shows that the sum of weighted point masses σ_c plugs the weakness in ν_c for $1\le t<\infty$.

Theorem 3.1. *For $1\le t<\infty$, $abpe(P^t(d\mu_{c,c})) = \mathbf{D}$.*

Proof. By Jensen's Inequality and since support$(\mu_{c,c}) \subseteq \overline{\mathbf{D}}$, we need only show that $\mathbf{D}\subseteq abpe(P^1(d\mu_{c,c}))$. Now for $N = 1,2,3,\dots$ let $r_N = 3cN$, let $W_N = \{\zeta : Re(\zeta)>0 \text{ and } |\zeta|<r_N\}$ and let $E_N = \{\zeta : Re(\zeta)\ge 0 \text{ and } |\zeta| = r_N\}$. Since $|\cos\zeta| \ge |\cos|\zeta||$ for all ζ in $\mathbf{C}$, we have, by our choice of r_N, that $|\cos(\frac{\pi}{3c}\zeta)|\ge 1$ for all ζ in E_N. Let $\Omega_N = \varphi^{-1}(W_N)$ ($\varphi(z) = \frac{1+z}{1-z}$) , let $\gamma_N = \varphi^{-1}(E_N)$ and let Γ_N denote $\partial\Omega_N$ parameterized once counterclockwise. Observe that Ω_N contains $\{z_n\}_{n=1}^N$ – the first N zeros of g_c in $\mathbf{D}$. So, by the Residue Theorem, if p is any polynomial, then

$$\frac{1}{2\pi i}\int_{\Gamma_N}\frac{p(z)}{g_c(z)}\frac{dz}{z} = \frac{p(0)}{g_c(0)} + \sum_{n=1}^{N}\frac{p(z_n)}{z_n g_c'(z_n)} .$$

Now for $N = 1,2,3,\dots$ and all z in γ_N , $|g_c(z)|\ge 1$. Moreover, dist$(0,\gamma_N)\to 1$ and length$(\gamma_N)\to 0$ as $N\to\infty$. Therefore,
$\frac{1}{2\pi}\int_{\gamma_N}\frac{|p(z)|}{|g_c(z)|}\frac{|dz|}{|z|}\to 0$ as $N\to\infty$ and hence

$$\begin{aligned} |p(0)| &\le \frac{1}{2\pi}\int_{\Gamma_N}\frac{|p(z)|}{|g_c(z)|}\frac{|dz|}{|z|} + \sum_{n=1}^{N}\frac{|p(z_n)|}{|z_n||g_c'(z_n)|} \\ &\to \int_{\partial\mathbf{D}}|p(z)|\frac{dm(z)}{|g_c(z)|} + \sum_{n=1}^{\infty}\frac{|p(z_n)|}{|z_n||g_c'(z_n)|} \text{ (as } N\to\infty) \\ &= ||p||_{L^1(d\mu_{c,c})}. \end{aligned}$$

Evidently, $0\in bpe(P^1(d\mu_{c,c}))$. So, by Lemma 2.1 and the fact that the components of $abpe(P^1(d\mu_{c,c}))$ are simply connected, we conclude that $\mathbf{D}\subseteq abpe(P^1(d\mu_{c,c}))$. □

The next theorem is our main result. It provides relatively simple examples of measures μ for which $abpe(P^t(d\mu))$ varies with t.

Theorem 3.2. *Let $1\le c\le d<\infty$. Then*
(1) $abpe(P^t(d\mu_{c,d})) = \emptyset$ if $1\le t<\frac{d}{c}$ and
(2) $abpe(P^t(d\mu_{c,d})) = \mathbf{D}$ if $\frac{d}{c}\le t<\infty$.

Proof. (1) Suppose $1 \le t < \frac{d}{c}$; and so $c < d$ here. For $n = 1, 2, 3, \ldots$ define f_n on $\{z : |z| < 1 + \frac{1}{n}\}$ by $f_n(z) = g_d(\frac{n}{n+1}z)$. Observe that $f_n \in P^t(d\mu_{c,d})$ and, by (3.1) and Lebesgue's Convergence Theorem, $f_n \to g_d$ in $L^t(d\mu_{c,d})$ as $n \to \infty$. Therefore, $g_d \in P^t(d\mu_{c,d})$. Now $|g_d|^t d\mu_{c,d} = |g_d|^t d\nu_c = \frac{|g_d|^t}{|g_c|} dm$ and since $1 \le t < \frac{d}{c}$ we have, by (3.1), that $\int \log(\frac{|g_d|^t}{|g_c|}) dm = -\infty$. Applying Szegö's Theorem, there is a sequence $\{p_n\}$ of polynomials such that $p_n(0) = 1$ for all n and yet $\int |p_n|^t |g_d|^t d\mu_{c,d} \to 0$ as $n \to \infty$. Therefore, $p_n g_d \in P^t(d\mu_{c,d})$ and $p_n(0)g_d(0) = g_d(0)\,(\ne 0)$ for all n and yet $||p_n g_d||_{L^t(d\mu_{c,d})} \to 0$ as $n \to \infty$. Consequently, $0 \notin bpe(P^t(d\mu_{c,d}))$ and therefore $abpe(P^t(d\mu_{c,d})) = \emptyset$.

(2) Suppose $\frac{d}{c} \le t < \infty$. By Jensen's Inequality we need only show that $abpe(P^t(d\mu_{c,d})) = \mathbf{D}$ for $t = \frac{d}{c}$. Now, by the proof of Theorem 3.1,

$$\begin{aligned} |p(0)| &\le ||p||_{L^1(d\mu_{d,d})} \\ &= \int_{\partial \mathbf{D}} \frac{|p(z)|}{|g_d(z)|} dm(z) + \int |p| d\sigma_d \end{aligned}$$

for any polynomial p. Therefore, by Jensen's Inequality and (3.1),

$$\begin{aligned} |p(0)|^{\frac{d}{c}} &\le c_1 \cdot \left(\int_{\partial \mathbf{D}} \left(\frac{|p(z)|}{|g_d(z)|} \right)^{\frac{d}{c}} dm(z) + \int |p|^{\frac{d}{c}} d\sigma_d \right) \\ &\le c_2 \cdot \left(\int_{\partial \mathbf{D}} |p(z)|^{\frac{d}{c}} d\nu_c(z) + \int |p|^{\frac{d}{c}} d\sigma_d \right), \end{aligned}$$

where c_1 and c_2 are constants that do not depend on p. Evidently $0 \in bpe(P^{\frac{d}{c}}(d\mu_{c,d}))$ and therefore $abpe(P^{\frac{d}{c}}(d\mu_{c,d})) = \mathbf{D}$. □

Remark 3.3. Our definition of g_c and hence our subsequent definitions of ν_c and σ_c were made for constants c that are greater than or equal to 1. This choice of c ensures that $g_c(0) \ne 0$, which facilitates some of our arguments − in particular, the application of the Residue Theorem in the proof of Theorem 3.1. By no means is this restriction on c essential. With mild adjustments in their proofs, Theorems 3.1 and 3.2 carry through for all $c > 0$.

Theorem 3.1 is the solution to a special case of a problem that can be posed in considerable generality. To describe this general problem we begin with a rectifiable Jordan arc γ having endpoints 0 and 1 such that $\gamma \backslash \{1\} \subseteq \mathbf{D}$. Let $\Omega = \mathbf{D} \backslash \gamma$ and let ω_Ω denote harmonic measure on $\partial\Omega$ evaluated at some point in Ω. Choose $h \ge 0$ in $L^\infty(dm)$ such that $\log(h) \in L^1(d\omega_\Omega|_{\partial\mathbf{D}})$; we may or may not have $\log(h)$ in $L^1(dm)$. Then one can find a function F that is analytic in Ω such that $F \circ \varphi$ (φ is a conformal mapping from $\mathbf{D}$ onto Ω) is an outer function and $|F|$ has well-defined boundary values equal to h *a.e.* ω_Ω on $\partial\mathbf{D}$ and equal to 1 *a.e.* ω_Ω on γ. If we let μ be the measure on $\partial\Omega$ given by $d\mu = h dm + d\omega_\Omega|_\gamma$, then we can use F to show that $\Omega \subseteq abpe(P^1(d\mu))$ and, since γ is rectifiable, by a standard argument we in fact get that $abpe(P^1(d\mu)) = \mathbf{D}$. So if $\log(h) \notin L^1(dm)$, then evidently $\omega_\Omega|_\gamma$ plugs the weakness in hdm. By [1], Theorem 1, we must have $\log(h)$ in $L^1(d\omega_\Omega|_{\partial\mathbf{D}})$, otherwise no finite, positive Borel measure on γ can plug the weakness in hdm. Can the role of $\omega_\Omega|_\gamma$ be assumed by a sum of weighted point masses that lie in γ? More explicitly, we are asking:

Question 3.4. Let γ and Ω be as in the above discussion, and suppose $0 \le h \in L^\infty(dm)$, $\log(h) \in L^1(d\omega_\Omega|_{\partial\mathbf{D}})$ and yet $\log(h) \notin L^1(dm)$. Does there exist a sequence $\{z_n\}$ in $\gamma\backslash\{1\}$ ($z_n \to 1$ as $n \to \infty$) and a summable sequence of positive constants $\{c_n\}$ such that if $\sigma := \sum_{n=1}^\infty c_n\delta_{z_n}$ and μ is given by $d\mu = hdm + d\sigma$, then $abpe(P^1(d\mu)) = \mathbf{D}$?

We first consider the case that $\gamma = [0,1]$. In this case, $\Omega = \mathbf{D}\backslash[0,1]$ and $d\omega_\Omega|_{\partial\mathbf{D}}$ is boundedly equivalent to $|z-1|dm(z)$. Therefore, if $1 \le a < 2$, $b > 0$ and $h(e^{i\theta}) := e^{-b\left(\frac{|\sin\theta|}{1-\cos\theta}\right)^a}$, then $h \in L^\infty(dm)$, $\log(h) \in L^1(d\omega_\Omega|_{\partial\mathbf{D}})$ and yet $\log(h) \notin L^1(dm)$. Theorem 3.1 (and Remark 3.3) give us a sum of weighted point masses σ that plugs the weakness in hdm for $a = 1$. For $1 < \lambda < 2$, let $\varphi_\lambda(z) = \left(\frac{1+z}{1-z}\right)^\lambda$ and let $g_{c,\lambda}(z) = \cos(\frac{\pi}{3c}\varphi_\lambda(z))$, where $c > 0$. If $1 < a < 2$ and $b > 0$, then an application of the Residue Theorem (involving $g_{c,a}$ for an appropriate choice of c), similar to that found in the proof of Theorem 3.1, provides a sum of weighted point masses σ that plugs the weakness in hdm for $h(e^{i\theta}) := e^{-b\left(\frac{|\sin\theta|}{1-\cos\theta}\right)^a}$. So the answer to Question 3.4 is in the affirmative if $\gamma = [0,1]$ and h has the form $h(e^{i\theta}) := e^{-b\left(\frac{|\sin\theta|}{1-\cos\theta}\right)^a}$, where $1 \le a < 2$ and $b > 0$. Moreover, $1 \le a < 2$ is the full range for which such measures hdm have a weakness that can be plugged by a measure with support in $[0,1]$ since, if $a \ge 2$, then $\log(h) \notin L^1(d\omega_\Omega|_{\partial\mathbf{D}})$. However, this particular collection of functions h by no means contains all of the boundary weights under consideration for $\gamma = [0,1]$. Therefore, Question 3.4 remains open even in the case that $\gamma = [0,1]$. For any arc γ that defines a nontangential approach to 1 in $\mathbf{D}$ the authors have been able to use the methods of Theorem 3.1 to answer Question 3.4 in the affirmative, though, again, only for a limited subcollection of the functions h under consideration for that γ.

By Theorem 3.2 part (1) we have that $abpe(P^1(d\mu_{1,2})) = \emptyset$ and so σ_2 fails to plug the weakness in ν_1 for $t = 1$; observe that the distribution of ν_1 on $\partial\mathbf{D}$ is symmetric with respect to the real line. We end this paper with a curious example which shows that this symmetry is important.

Example 3.5. Let

$$f(\zeta) = e^{-\frac{i\pi}{6}\zeta} \cdot \cos\left(\frac{\pi}{6}\zeta\right) \quad \left(= \frac{1}{2}(1 + e^{-\frac{i\pi}{3}\zeta})\right)$$

and let $g(z) = f(\varphi(z))$, where $\varphi(z) = \frac{1+z}{1-z}$. Define ν on $\partial\mathbf{D}$ by $d\nu = \frac{1}{|g|}dm$, and let $\mu = \nu + \sigma_2$. Arguing as in the proof of Theorem 3.1, with g in place of g_c, we get that $abpe(P^1(d\mu)) = \mathbf{D}$ and so σ_2 plugs the weakness in ν for $t = 1$. Observe that ν is boundedly equivalent to ν_1 on $\{z : |z| = 1 \text{ and } Im(z) \ge 0\}$, though ν is boundedly equivalent to m on $\{z : |z| = 1 \text{ and } Im(z) \le 0\}$.

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