PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 2, February 1999, Pages 455–464
S 0002-9939(99)04638-9

UNCONDITIONAL BASIC SEQUENCE IN $L^p(\mu)$ AND ITS l^p-STABILITY

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(Communicated by J. Marshall Ash)

ABSTRACT. This paper is concerned with unconditional basic sequences in $L^p(\mu)$. We prove that, under some conditions, a sequence in $L^p(\mu)$ is a bounded unconditional basic sequence if and only if it is l^p-stable. At last the results are applied to the shift-invariant basic sequences generated by a finite subset of $L^p(R^s)$, which is very important in wavelet analysis.

1. INTRODUCTION

Unconditional basic sequences are very important in the basis theory of Banach spaces. It is a motivation for the development of the recent wavelet analysis. In general, it is difficult to verify whether a given basic sequence is unconditional. Therefore, it is meaningful to find more practical but equivalent conditions. In $L^2(\mu)$ it is well known that a sequence is a bounded unconditional basic sequence if and only if it is l^2-stable. We want to know whether the result can be extended to $L^p(\mu)$ $(1 \le p \le \infty)$. In this paper, we mainly study some properties of unconditional basic sequences and then prove that, under some conditions, a bounded basic sequence in $L^p(\mu)$ is unconditional if and only if it is l^p-stable. Finally, the results are applied to the shift-invariant basic sequence generated by a finite subset of $L^p(R^s)$, which is very important in wavelet analysis.

Let (Ω, Σ, μ) be a σ-finite (positive) measure space, $L^p(\mu) := L^p(\Omega, \Sigma, \mu)$ $(1 \le p \le \infty)$ the space of all p-integrable functions on (Ω, Σ, μ) with norm defined by

$$\|f\|_p := \begin{cases} (\int_\Omega \mid f(x) \mid^p d\mu(x))^{1/p}, & f \in L^p(\mu)\ (1 \le p < \infty), \\ ess\ sup_{x\in\Omega} \mid f(x) \mid, & f \in L^\infty(\mu)\ (p = \infty). \end{cases}$$

As usual, for $1 \le p \le \infty$, $l^p := l^p(\mathbf{N})$ ($\mathbf{N}$ denotes the set of all natural numbers) denotes the space of all sequences a such that $\|a\|_p < \infty$, where

$$\|a\|_p := \begin{cases} (\sum_{j=1}^\infty \mid a_j \mid^p)^{1/p}, & 1 \le p < \infty, \\ \sup_{j\in\mathbf{N}} \mid a_j \mid, & p = \infty. \end{cases}$$

For simplicity, let $l := l(\mathbf{N})$ denote the space of all sequences and

$$l_0 := l_0(\mathbf{N}) := \{a \in l \mid a_j \ne 0 \text{ for only finite } j\}.$$

Received by the editors October 14, 1996 and, in revised form, May 21, 1997.
1991 *Mathematics Subject Classification.* Primary 46B20.
Key words and phrases. Unconditional basic sequence, l^p-stability, wavelet analysis.

Let X be a Banach space. A sequence $\{x_j\}_{j=1}^\infty \subset X$ is called a (Schauder) basis if $\forall x \in X$, there exists unique sequence $a \in l$ such that $x = \sum_{j=1}^\infty a_j x_j$.

A Schauder basis $\{x_j\}_{j=1}^\infty$ of a Banach space X is called an unconditional basis if $\forall \{a_j\}_{j=1}^\infty \in l$, the convergence of $\sum_{j=1}^\infty a_j x_j$ in X implies $\sum_{j=1}^\infty \theta_j a_j x_j$ converges in X provided $\theta_j = \pm 1$; a sequence $\{x_j\}_{j=1}^\infty$ is called a (or: an unconditional) basic sequence in a Banach space X if it is a (or: an unconditional) basis of a closed subspace of X.

A sequence $\{x_j\}_{j=1}^\infty \subset X$ is called bounded if

$$0 < \inf_{j\in\mathbf{N}} \|x_j\| \le \sup_{j\in\mathbf{N}} \|x_j\| < \infty.$$

$\{x_j\}_{j=1}^\infty \subset X$ is called l^p-stable ($1 \le p \le \infty$) if there exist positive constants c_p and C_p such that $\forall a \in l_0$,

$$c_p \|a\|_p \le \|\sum_{j=1}^\infty a_j x_j\| \le C_p \|a\|_p$$

i.e., simply

$$\|\sum_{j=1}^\infty a_j x_j\| \sim \|a\|_p.$$

Hereafter, $A \sim B$ means there exist positive constants c, C such that $cA \le B \le CA$. The preceding definitions can be found in [SYL], [JM] and [JIA].

The paper is organized as follows. In section 2, we give an intuitive characterization and an equivalent norm for unconditional basic sequences. In section 3 the equivalence between l^p-stability and unconditionality is studied. Finally, the shift-invariant family generated by a finite set of $L^p(R^s)$ is studied in section 4.

2. Unconditional basic sequence in $L^p(\mu)$

The classical theory of unconditional bases in general Banach spaces contains rich contents. Some sufficient and necessary conditions for unconditional bases and their corresponding properties have been established (see [SYL]). In this paper, we prove the following theorem at first, which gives an intuitive characterization on unconditional basic sequences.

Theorem 1. *A basic sequence $\{x_n\}_{n=1}^\infty$ in a Banach space is unconditional if and only if, for any two disjoint finite subsets A, B of $\mathbf{N}$, one of the following conditions holds:*

(i) $\|x + y\| \sim \|x - y\| \qquad \forall x \in span\{x_j \mid j \in A\},\ y \in span\{x_j \mid j \in B\}$.

(ii) There exists a constant $C > 0$ such that $\|x - y\| \ge C$ for any $x \in span\{x_j \mid j \in A\}$, $y \in span\{x_j \mid j \in B\}$ with $\|x\| = \|y\| = 1$.

Proof. Let $\{x_n\}$ be an unconditional basis of Banach space X and A, B be two disjoint finite subsets of $\mathbf{N}$. Then there exists a constant $K > 0$ such that (see [SYL])

$$\|y\| \le K\|x + y\| \qquad \forall x \in span\{x_j \mid j \in A\},\ y \in span\{x_j \mid j \in B\}.$$

Hence

$$\|x - y\| = \|(x + y) - 2y\| \le (1 + 2K)\|x + y\|.$$

The inverse inequality can be proved similarly. (i) follows.

In the sequel, we prove (ii); we assume (i) holds now.

$\forall x \in span\{x_j \mid j \in A\}$, $y \in span\{x_j \mid j \in B\}$ with $\|x\| = \|y\| = 1$, it is easily seen that $\|x+y\| \geq \frac{1}{2}$ or $\|x-y\| \geq \frac{1}{2}$. If $\|x-y\| \geq \frac{1}{2}$, (ii) holds obviously; if $\|x+y\| \geq \frac{1}{2}$, then $\|x-y\| \sim \|x+y\| \geq \frac{1}{2}$, i.e. (ii) also holds.

Finally, let (ii) hold. We prove that $\{x_n\}$ is unconditional.

Let $x \in span\{x_j \mid j \in A\}$, $y \in span\{x_j \mid j \in B\}$ with $x \neq 0,\ y \neq 0$. Then there exists $C > 0$ such that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geq C.$$

Choosing $\sigma > 0$ satisfying $0 < 2(1-\sigma) \leq C$ and assuming $\|y\| \leq \|x\|$ without loss of generality, we have that

(1^0). If $\|y\| \geq \sigma\|x\|$, then

$$\begin{aligned} \|x-y\| &= \left\| \frac{x}{\|x\|}\|x\| - \frac{y}{\|y\|}\|x\| + \frac{y}{\|y\|}\|x\| - y \right\| \\ &\geq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \cdot \|x\| - \left| \frac{\|x\|}{\|y\|} - 1 \right| \cdot \|y\| \\ &\geq C\|x\| - (\|x\| - \|y\|) \\ &\geq \frac{1}{2}C\|x\|. \end{aligned}$$

Hence

$$\frac{\|x+y\|}{\|x-y\|} \leq \frac{\|x\| + \|y\|}{\frac{1}{2}C\|x\|} \leq \frac{4}{C}.$$

(2^0). If $\|y\| \leq \sigma\|x\|$, one easily gets

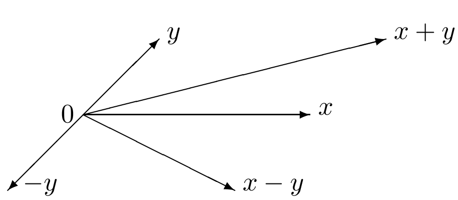
$$\frac{\|x+y\|}{\|x-y\|} \leq \frac{2\|x\|}{\|x\| - \|y\|} \leq \frac{2}{1-\sigma}.$$

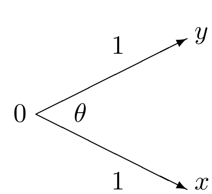
In any case, there always exists $K > 0$ such that $\|x+y\| \leq K\|x-y\|$. Hence

$$\|x\| \leq \frac{1}{2}\|x + (-y)\| + \frac{1}{2}\|x - (-y)\| \leq \frac{1}{2}(1+K)\|x+y\|$$

which implies that $\{x_n\}$ is an unconditional basic sequence (see [SYL]). Theorem 1 follows. □

Remark. We give an intuitive explanation of the theorem as follows. A basic sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space is unconditional if and only if, for any two disjoint finite subsets $A,\ B$ of $\mathbf{N}$, the included angle between the subspaces $span\{x_j \mid j \in A\}$ and $span\{x_j \mid j \in B\}$ is larger than a positive number independent of A and B.





Theorem 2. *Let $\{x_j\}_{j=1}^{\infty}$ be a basic sequence in $L^p(\mu)$ $(1 \le p \le \infty)$. Then it is unconditional if and only if*

$$\|\sum_{j=1}^{\infty} a_j x_j\|_p \sim \|(\sum_{j=1}^{\infty} | a_j x_j |^2)^{1/2}\|_p \qquad \forall x = \sum_{j=1}^{\infty} a_j x_j \in L^p(\mu).$$

Proof. We only need to prove the necessity. In fact, $\forall x = \sum_{j=1}^{\infty} a_j x_j \in X$, by Khintchine's inequality (see [SYL, p.30]) we have that

$$\int_0^1 | \sum_{j=1}^{m} r_j(t) a_j x_j(\omega) |^p \, dt \sim (\sum_{j=1}^{m} | a_j x_j(\omega) |^2)^{p/2} \qquad \forall \omega \in \Omega,\ m \in \mathbf{N}$$

where $\{r_j(t)\}$ are the Rademacher functions, i.e.

$$r_j(t) := signsin(2^j \pi t) \qquad (j = 1, 2, \cdots).$$

Hence

$$\int_0^1 \|\sum_{j=1}^{m} r_j(t) a_j x_j\|_p^p dt = \int_0^1 dt \int_\Omega | \sum_{j=1}^{m} r_j(t) a_j x_j(\omega) |^p \, d\omega$$
$$\sim \int_\Omega (\sum_{j=1}^{m} | a_j x_j(\omega) |^2)^{p/2} d\mu(\omega).$$

Since $\{x_j\}$ is an unconditional basic sequence and $r_j(t) = \pm 1$ *a.e.*, we have (see [SYL, p.22])

$$\|\sum_{j=1}^{m} a_j x_j\|_p^p = \int_0^1 \|\sum_{j=1}^{m} a_j x_j\|_p^p dt \sim \int_0^1 \|\sum_{j=1}^{m} r_j(t) a_j x_j\|_p^p dt$$
$$\sim \int_\Omega (\sum_{j=1}^{m} | a_j x_j(\omega) |^2)^{p/2} d\mu(\omega).$$

Letting $m \to \infty$ we deduce that

$$\|\sum_{j=1}^{\infty} a_j x_j\|_p \sim \|(\sum_{j=1}^{\infty} | a_j x_j |^2)^{1/2}\|_p.$$

This completes the proof. ☐

Remark. If X is the closed subspace of $L^p(\mu)$ such that $\{x_j\}_{j=1}^{\infty}$ is its basis, it is clear that

$$|\!|\!| \sum_{j=1}^{\infty} a_j x_j |\!|\!|_p := \|(\sum_{j=1}^{\infty} | a_j x_j |^2)^{1/2}\|_p \qquad (\forall x = \sum_{j=1}^{\infty} a_j x_j \in X)$$

is a norm on X.

3. l^p-STABILITY AND UNCONDITIONAL BASIC SEQUENCES IN $L^p(\mu)$

In this section, we study the equivalence between the l^p-stability and unconditionality of bounded basic sequences in $L^p(\mu)$. Our results show that, under some conditions, $\{x_j\}$ is a bounded unconditional basic sequence if and only if it is l^p-stable.

Theorem 3. *Let $\{x_j\}_{j=1}^{\infty}$ be a l^p-stable $(1 \le p \le \infty)$ sequence in a Banach space X. Then it is a bounded unconditional basic sequence in X.*

Proof. $\forall x \in \overline{span\{x_j\}}$(the closure in X) and $n \in \mathbf{N}$, let $a^{(n)} \in l_0$ such that

$$\|x - \sum_{j=1}^{\infty} a_j^{(n)} x_j\| < \frac{1}{n}.$$

It is easily seen that

$$\|a^{(n)} - a^{(m)}\|_p \sim \|\sum_{j=1}^{\infty}(a_j^{(n)} - a_j^{(m)})x_j\| \longrightarrow 0 \qquad (n,\ m \to \infty).$$

Hence there exists $a \in l^p$ such that

$$\|a^{(n)} - a\|_p \longrightarrow 0 \qquad (n \to \infty).$$

For any $n \in \mathbf{N}$, let $m \geq k$ be large enough such that $a_j^{(n)} = 0\ (\forall j \geq k)$. By the l^p-stability, we get

$$\|\sum_{j=k}^{m} a_j x_j\| = \|\sum_{j=k}^{m}(a_j - a_j^{(n)})x_j\| \leq C_p\|a - a^{(n)}\|_p;$$

therefore, $\sum_{j=1}^{\infty} a_j x_j$ converges in X. It is also clear that

$$\|\sum_{j=1}^{\infty} a_j^{(n)} x_j - \sum_{j=1}^{\infty} a_j x_j\| \sim \|a^{(n)} - a\|_p \longrightarrow 0\ \ (n \to \infty).$$

Hence $x = \sum_{j=1}^{\infty} a_j x_j$. Again by the l^p-stability we conclude that such expression is unique, thus $\{x_j\}_{j=1}^{\infty}$ is a Schauder basis of $\overline{span\{x_j\}}$. At last, the l^p-stability implies that it is unconditional and bounded obviously. The proof of the theorem is complete. □

Theorem 4. *Let $\{x_j\}_{j=1}^{\infty}$ be a bounded unconditional basic sequence in $L^p(\mu)$ $(1 \leq p \leq \infty)$. Then there exist positive constants c_p, C_p such that*

(i) if $1 \leq p \leq 2$, there holds

$$c_p\|\{a_j\}\|_2 \leq \|\sum_{j=1}^{\infty} a_j x_j\|_p \leq C_p\|\{a_j\}\|_p \qquad \forall \sum_{j=1}^{\infty} a_j x_j \in L^p(\mu);$$

(ii) if $2 \leq p \leq \infty$, there holds

$$c_p\|\{a_j\}\|_p \leq \|\sum_{j=1}^{\infty} a_j x_j\|_p \leq C_p\|\{a_j\}\|_2 \qquad \forall \sum_{j=1}^{\infty} a_j x_j \in L^p(\mu).$$

Proof. (i). For $1 \leq p \leq 2$, we know that $L^p(\mu)$ is of Rademacher type p and Rademacher cotype 2 (see [SYL, pp.151–152]), i.e. there exist $c_p,\ C_p > 0$ such that

$$(\int_0^1 \|\sum_{j=1}^{\infty} r_j(t) z_j\|_p^2 dt)^{1/2} \leq C_p(\sum_{j=1}^{\infty} \|z_j\|_p^p)^{1/p}$$

and

$$(\int_0^1 \|\sum_{j=1}^{\infty} r_j(t) z_j\|_p^2 dt)^{1/2} \geq c_p(\sum_{j=1}^{\infty} \|z_j\|_p^2)^{1/2}$$

for all $n \in \mathbf{N}$ and $\{z_j\}_{j=1}^{\infty} \subset L^p(\mu)$, where $\{r_j(t)\}$ are the Rademacher functions. Since $\{x_j\}_{j=1}^{\infty}$ is an unconditional basic sequence in $L^p(\mu)$, we conclude that

$$\begin{aligned}\|\sum_{j=1}^{\infty} a_j x_j\|_p &\sim (\int_0^1 \|\sum_{j=1}^{\infty} r_j(t) a_j x_j\|_p^2 dt)^{1/2} \\ &\le C_p(\sum_{j=1}^{\infty} \|a_j x_j\|_p^p)^{1/p} \\ &\le C_p(\sum_{j=1}^{\infty} \mid a_j \mid^p)^{1/p} \sup_{j\in\mathbf{N}} \|x_j\|_p\end{aligned}$$

and

$$\begin{aligned}\|\sum_{j=1}^{\infty} a_j x_j\|_p &\sim (\int_0^1 \|\sum_{j=1}^{\infty} r_j(t) a_j x_j\|_p^2 dt)^{1/2} \\ &\ge c_p(\sum_{j=1}^{\infty} \|a_j x_j\|_p^2)^{1/2} \\ &\ge c_p(\sum_{j=1}^{\infty} \mid a_j \mid^2)^{1/2} \inf_{j\in\mathbf{N}} \|x_j\|_p.\end{aligned}$$

This completes the proof of (i).

(ii). For $2 \le p < \infty$, the proof is similar. We consider the case of $p = \infty$ at last. By Theorem 2, it is easily seen that

$$\|\sum_{j=1}^{\infty} a_j x_j\|_\infty \sim \|(\sum_{j=1}^{\infty} \mid a_j x_j \mid^2)^{1/2}\|_\infty \le (\sum_{j=1}^{\infty} \mid a_j \mid^2)^{1/2} \sup_{j\in\mathbf{N}} \|x_j\|_\infty$$

and

$$\|\sum_{j=1}^{\infty} a_j x_j\|_\infty \sim \|(\sum_{j=1}^{\infty} \mid a_j x_j \mid^2)^{1/2}\|_\infty \ge \sup_{j\in\mathbf{N}} \|a_j x_j\|_\infty \ge \|\{a_j\}\|_\infty \inf_{j\in\mathbf{N}} \|x_j\|_\infty.$$

Hence the theorem also holds for $p = \infty$. The proof is complete. □

Theorem 5. *Let $\{x_j\}_{j=1}^{\infty}$ be a bounded unconditional basic sequence in $L^p(\mu)$ $(1 \le p \le \infty)$ and*

$$\tilde{x}_j := \mid x_j \mid^{2/p} (\sum_{i=1}^{\infty} \mid x_i \mid^2)^{\frac{1}{2}-\frac{1}{p}} \qquad (1 \le p \le \infty)$$

where $\tilde{x}_j := 0$ if $\sum_{i=1}^{\infty} \mid x_i \mid = 0$ and $\frac{1}{p} := 0$ for $p = \infty$. If

$$0 < \inf_{j\in\mathbf{N}} \|\tilde{x}_j\|_p \le \sup_{j\in\mathbf{N}} \|\tilde{x}_j\|_p < \infty$$

then $\{x_j\}_{j=1}^{\infty}$ is l^p-stable.

Proof. (1). If $1 \le p < 2$, by Theorem 4 and Theorem 2 we have that, $\forall x = \sum_{j=1}^{\infty} a_j x_j \in L^p(\mu)$, there exists $C_p > 0$ such that

$$\|x\|_p \le C_p \|\{a_j\}\|_p$$

and

$$\|x\|_p \sim (\int_\Omega (\sum_{j=1}^{\infty} | a_j x_j |^2)^{p/2} d\mu)^{1/p}.$$

Using the inverse Hölder-inequality, we have that, for $\theta := 2/p$,

$$\begin{aligned}(\sum_{j=1}^{\infty} | a_j x_j |^2)^{p/2} &= (\sum_{j=1, x_j\neq 0}^{\infty} | a_j |^2 | x_j |^{2\theta} | x_j |^{2(1-\theta)})^{p/2} \\ &\geq [(\sum_{j=1, x_j\neq 0}^{\infty} (| a_j |^2 | x_j |^{2\theta})^{p/2})^{2/p} (\sum_{j=1, x_j\neq 0}^{\infty} | x_j |^{2(1-\theta)\frac{p}{p-2}})^{1-\frac{2}{p}}]^{p/2} \\ &= (\sum_{j=1, x_j\neq 0}^{\infty} | a_j |^p | x_j |^{\theta p})(\sum_{j=1, x_j\neq 0}^{\infty} | x_j |^{2(1-\theta)\frac{p}{p-2}})^{\frac{p}{2}-1} \\ &= \sum_{j=1, x_j\neq 0}^{\infty} | a_j |^p | x_j |^{\theta p} (\sum_{j=1, x_j\neq 0}^{\infty} | x_j |^{2(1-\theta)\frac{p}{p-2}})^{\frac{p}{2}-1} \\ &= \sum_{j=1}^{\infty} | a_j |^p |\tilde{x}_j|^p.\end{aligned}$$

Hence

$$\begin{aligned}\|x\|_p &\sim (\int_\Omega (\sum_{j=1}^{\infty} | a_j x_j |^2)^{p/2} d\mu)^{1/p} \\ &\geq (\int_\Omega \sum_{j=1}^{\infty} | a_j |^p |\tilde{x}_j|^p d\mu)^{1/p} \geq (\sum_{j=1}^{\infty} | a_j |^p)^{1/p} \inf_{j\in \mathbf{N}} \|\tilde{x}_j\|_p,\end{aligned}$$

i.e. $\|x\|_p \sim \|\{a_j\}\|_p$.

(2). If $2 < p < \infty$, we can conclude by the same way that, $\forall x = \sum_{j=1}^{\infty} a_j x_j \in L^p(\mu)$, there exists $c_p > 0$ such that

$$\|x\|_p \geq c_p \|\{a_j\}\|_p$$

and, for $\theta := 2/p$,

$$\begin{aligned}\|x\|_p &\sim (\int_\Omega (\sum_{j=1}^{\infty} | a_j x_j |^2)^{p/2} d\mu)^{1/p} \\ &= [\int_\Omega (\sum_{j=1}^{\infty} | a_j |^2 | x_j |^{2\theta} | x_j |^{2(1-\theta)})^{p/2} d\mu]^{1/p} \\ &\leq [\int_\Omega (\sum_{j=1}^{\infty} | a_j |^p | x_j |^{\theta p})(\sum_{j=1}^{\infty} | x_j |^{2(1-\theta)\frac{p}{p-2}})^{\frac{p}{2}-1} d\mu]^{1/p} \\ &= [\sum_{j=1}^{\infty} | a_j |^p (\int_\Omega |\tilde{x}_j|^p d\mu)]^{1/p} \\ &\leq (\sum_{j=1}^{\infty} | a_j |^p)^{1/p} \sup_{j\in \mathbf{N}} \|\tilde{x}_j\|_p.\end{aligned}$$

Hence $\|x\|_p \sim \|\{a_j\}\|_p$.

(3). If $p = 2$, the proof is trivial.

(4). At last, if $p = \infty$, it is easily seen by Theorem 4 that

$$\|x\| \geq c_p \|\{a_j\}\|_\infty$$

and

$$\|x\|_p \sim \|(\sum_{j=1}^{\infty} \mid a_j x_j \mid^2)^{1/2}\|_\infty \leq \|\{a_j\}\|_\infty \|(\sum_{j=1}^{\infty} \mid x_j \mid^2)^{1/2}\|_\infty$$

i.e. $\|x\|_p \sim \|\{a_j\}\|_p$. The proof of the theorem is complete. □

Remark. From the proof above, we can see easily that, for $1 \leq p < 2$ and $2 < p < \infty$, Theorem 5 holds still if $\tilde{x}_j$ is replaced by the following function:

$$g_{j,\theta} := |x_j|^\theta (\sum_{i=1}^{\infty} |x_i|^{\frac{2p(1-\theta)}{p-2}})^{\frac{1}{2}-\frac{1}{p}}$$

where θ is any positive constant independent of j satisfying $\frac{2p(1-\theta)}{p-2} > 0$. It is clear that $g_{j,\theta} = \tilde{x}_j$ if $\theta := 2/p$.

4. Shift-invariant subspace of $L^p(R^s)$

Let $L^p(\mu) = L^p(R^s)$ ($s \in \mathbf{N}, 1 \leq p \leq \infty$) be the space of p-power Lebesgue integrable functions. For a finite subset $\Phi \subset L^p(R^s)$, the set generated by the shifts of Φ is defined by

$$S(\Phi) := \{\phi(\cdot - j) \mid j \in \mathbf{Z}^s,\ \phi \in \Phi\}$$

where $\mathbf{Z}$ is the set of all integers. The l^p-stability and unconditionality of $S(\Phi)$ are important topics in wavelet analysis. In this field, many interesting results have been established (see [MEYER, p.30], [DAU, p.298], [JM], etc.). In this section, we will use Theorem 5 to show that $S(\Phi)$ is l^p-stable if and only if it is a bounded unconditional basic sequence under some weak conditions.

We recall the space $\mathcal{L}^p$ defined in [JM] by

$$\mathcal{L}^p := \mathcal{L}^p(R^s) := \{f \mid f \text{ is Lebesgue measurable and } \mid f \mid_p := \|f^0\|_{L^p([0,1)^s)} < \infty\}$$

where

$$f^0 := \sum_{\alpha \in \mathbf{Z}^s} \mid f(\cdot - \alpha) \mid .$$

For convenience, we define another space as follows:

$$\mathcal{L}^p_* := \mathcal{L}^p_*(R^s) := \{f \mid f \text{ is Lebesgue measurable and } \mid f \mid^*_p := \|f^*\|_{L^p([0,1)^s)} < \infty\}$$

where

$$f^* := (\sum_{\alpha \in \mathbf{Z}^s} \mid f(\cdot - \alpha) \mid^2)^{1/2}.$$

It is easy to verify that

$$\begin{cases} \mathcal{L}^p \subset L^p \subset \mathcal{L}^p_* & (1 \leq p \leq 2), \\ \mathcal{L}^p \subset \mathcal{L}^p_* \subset L^p & (2 \leq p \leq \infty). \end{cases}$$

Theorem 6. *Let $S(\Phi)$ be generated by the shifts of finite subset $\Phi \subset L^p(R^s) \cap \mathcal{L}_*^p(R^s)$ $(1 \le p \le \infty,\ s \in \mathbf{N})$. Then $S(\Phi)$ is an unconditional basic sequence in $L^p(R^s)$ if and only if it is l^p-stable, i.e.*

$$\| \sum_{j\in\mathbf{Z}^s,\phi\in\Phi} a_j \phi(\cdot - j)\| \sim \|\{a_j\}\|_p \qquad \forall \sum_{j\in\mathbf{Z}^s,\phi\in\Phi} a_j\phi(\cdot - j) \in L^p(R^s).$$

Proof. We only need to prove the necessity. Let $S(\Phi)$ be an unconditional basic sequence in $L^p(\mu)$. $\forall \phi \in \Phi$ and $j \in \mathbf{Z}^s$, let

$$\tilde{\phi}(\cdot - j) :=\mid \phi(\cdot - j) \mid^{2/p} (\sum_{j\in\mathbf{Z}^s,\psi\in\Phi} \mid \psi(\cdot - j) \mid^2)^{\frac{1}{2}-\frac{1}{p}}.$$

One has

$$\|\phi(\cdot - j)\|_p = \|\phi\|_p$$

and

$$\begin{aligned} \|\tilde{\phi}(\cdot - j)\|_p &= (\int_{R^s} \mid \phi(x-j) \mid^2 (\sum_{i\in\mathbf{Z}^s,\psi\in\Phi} \mid \psi(x-i) \mid^2)^{\frac{p}{2}-1} dx)^{1/p} \\ &= (\int_{R^s} \mid \phi(x) \mid^2 (\sum_{i\in\mathbf{Z}^s,\psi\in\Phi} \mid \psi(x-i) \mid^2)^{\frac{p}{2}-1} dx)^{1/p} \\ &= (\int_{[0,1)^s} (\sum_{i\in\mathbf{Z}^s} \mid \phi(x-i) \mid^2)(\sum_{i\in\mathbf{Z}^s,\psi\in\Phi} \mid \psi(x-i) \mid^2)^{\frac{p}{2}-1} dx)^{1/p}. \end{aligned}$$

Hence

$$\begin{aligned} \|\tilde{\phi}(\cdot - j)\|_p &\le (\int_{[0,1)^s} (\sum_{i\in\mathbf{Z}^s,\psi\in\Phi} \mid \psi(x-i) \mid^2)^{p/2} dx)^{1/p} \\ &\le (\int_{[0,1)^s} (\sum_{\psi\in\Phi} (\sum_{i\in\mathbf{Z}^s} \mid \psi(x-i) \mid^2)^{1/2})^p dx)^{1/p} \\ &\le \sum_{\psi\in\Phi} (\int_{[0,1)^s} (\sum_{i\in\mathbf{Z}^s} \mid \psi(x-i) \mid^2)^{p/2} dx)^{1/p} \\ &= \sum_{\psi\in\Phi} \mid \psi \mid_p^* \end{aligned}$$

and

$$\begin{aligned} \|\tilde{\phi}(\cdot - j)\|_p &\ge (\int_{[0,1)^s} (\sum_{i\in\mathbf{Z}^s} \mid \phi(x-i) \mid^2)^{p/2} dx)^{1/p} \\ &= \mid \phi \mid_p^* . \end{aligned}$$

Therefore

$$\min_{\phi\in\Phi} \|\phi\|_p = \inf_{j\in\mathbf{Z}^s,\phi\in\Phi} \|\phi(\cdot - j)\|_p \le \sup_{j\in\mathbf{Z}^s,\phi\in\Phi} \|\phi(\cdot - j)\|_p = \max_{\phi\in\Phi} \|\phi\|_p$$

and

$$\min_{\phi\in\Phi} \mid \phi \mid_p^* \le \inf_{j\in\mathbf{Z}^s,\phi\in\Phi} \|\tilde{\phi}(\cdot - j)\|_p \le \sup_{j\in\mathbf{Z}^s,\phi\in\Phi} \|\tilde{\phi}(\cdot - j)\|_p \le \sum_{\phi\in\Phi} \mid \phi \mid_p^* .$$

Since $S(\Phi)$ is a basic sequence in $L^p(R^s)$, we conclude that $\min_{\phi\in\Phi} \|\phi\|_p > 0$ and $\min_{\phi\in\Phi} \mid \phi \mid_p^* > 0$. Moreover, $\Phi \subset L^p(R^s) \cap \mathcal{L}_*^p(R^s)$ implies $\max_{\phi\in\Phi} \|\phi\|_p < \infty$ and $\sum_{\phi\in\Phi} \mid \phi \mid_p^* < \infty$. Then, the conclusion follows from Theorem 5. □

By [JIA, Theorem 1.1–1.2] we further obtain the following theorem.

Theorem 7. *Let* $S(\Phi)$ *be generated by the finite subset* $\Phi \subset \mathcal{L}^p(R^s)$ $(1 \leq p \leq \infty,\ s \in \mathbf{N})$. *Then the following conditions are equivalent:*

(1) $S(\Phi)$ *constitutes an unconditional basic sequence in* $L^p(R^s)$,

(2) $S(\Phi)$ *is* l^p*-stable,*

(3) $\{\hat{\phi}(\xi + 2\pi\beta)\}_{\phi\in\Pi,\beta\in\mathbf{Z}^s}$ *are linearly independent,*

where $\hat{\phi}$ *is the Fourier transform of* ϕ *defined in* [JIA].

For the definition of "linear independent", we refer to [JIA].

Acknowlegement

The author is grateful to Professor J. Marshall Ash and the referee for their helpful suggestions.

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