PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 2, February 1999, Pages 327–336
S 0002-9939(99)04744-9

GROUP ALGEBRAS WITH UNITS SATISFYING A GROUP IDENTITY

CHIA-HSIN LIU

(Communicated by Ken Goodearl)

ABSTRACT. Let $K[G]$ be the group algebra of a group G over a field K, and let $U(K[G])$ be its group of units. A conjecture by Brian Hartley asserts that if G is a torsion group and $U(K[G])$ satisfies a group identity, then $K[G]$ satisfies a polynomial identity. This was verified earlier in case K is an infinite field. Here we modify the original proof so that it handles fields of all sizes.

1. INTRODUCTION

Let $K[G]$ be the group algebra of a torsion group G over an arbitrary field K. The group of units $U = U(K[G])$ is said to satisfy a group identity if there exists a nontrivial word $w = w(x_1, \dots, x_m)$ in the free group generated by $x_1, \dots, x_m$ such that $w(u_1, \dots, u_m) = 1$ for all $u_i \in U$. The goal of this paper is to prove

Theorem 1.1. *Let K be a field and G a torsion group. If $U(K[G])$ satisfies a group identity, then $K[G]$ satisfies a polynomial identity.*

This theorem was originally conjectured by Brian Hartley and first studied by D. S. Warhurst in the early 80s. For any infinite field K, J. Z. Gonçalves and A. Mandel [GM91] settled a special case, namely, when $U(K[G])$ satisfies a semigroup identity. A. Giambruno, E. Jespers and A. Valenti [GJV94] settled the case when G has no p-element if $char K = p$. Later, A. Giambruno, S. Sehgal and A. Valenti [GSV] confirmed Hartley's conjecture in case K is infinite. With this result, D. S. Passman [Pas] obtained necessary and sufficient conditions on the group G for $U(K[G])$ to satisfy a group identity. Furthermore, Y. Billig, D. Riley and V. Tasić [BRT] proved that $U(K[G])$ satisfies a group identity if and only if $K[G]$ satisfies a non-matrix polynomial identity, i.e., a polynomial identity not satisfied by the algebra of 2×2 matrices.

Almost all these works assumed that the field K is infinite so that they could apply a "Vandermonde determinant argument". One example is [GJV94, Proposition 1], which basically says that if a, b are in an algebra whose units satisfy a group identity and $a^2 = b^2 = 0$, then $(ab)^m = 0$ for some integer m determined by the identity. This proposition and its corollaries played an important role in [GJV94], [GSV], [Pas] and [BRT]. It seems that an analogue for algebras over a finite field is necessary. But this proposition is certainly false in this case. Consider $M_2(K)$, the algebra of 2×2 matrices over a finite field K, $a = e_{12} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $b = e_{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$;

Received by the editors March 25, 1997 and, in revised form, September 29, 1997.
1991 *Mathematics Subject Classification.* Primary 16S34.

then $U(M_2(K))$ satisfies a group identity since it is a finite group and $a^2 = b^2 = 0$, but $ab = e_{11}$ is not nilpotent. Thus it is natural to look for a weaker conclusion and this is done in Lemma 3.1 which shows that ab satisfies a polynomial determined by the group identity whether K is infinite or finite.

Another important lemma when K is infinite is [GSV, Lemma 2.3] which says that for any nonabelian finite group G, $U(K[G])$ satisfies a group identity if and only if G is p-abelian and $char K = p$. This is no longer true when K is finite since for any finite group G which is not p-abelian, $U(K[G])$ is finite and hence satisfies a group identity. Fortunately, we are able to avoid using this lemma in our proof.

This paper is organized as follows. In section 2, we give some definitions, make some conventions, prove some basic lemmas and recall some classical results. In section 3, we show that if $U(K[G])$ satisfies a group identity and $K[G]$ satisfies a nondegenerate generalized polynomial identity, then $K[G]$ satisfies a polynomial identity. Finally, we prove Theorem 1.1 in section 4.

2. Preliminaries

Throughout this paper, we use w to be a reduced word in the free group, and $w = 1$ to be a group identity. If a group identity is given, the following well-known lemma shows that we can actually construct a group identity in two variables of a form we like.

Lemma 2.1. *Let R be a ring. If $U(R)$ satisfies the group identity $w = 1$, then $U(R)$ satisfies a group identity of the following form:*

$$w_0(x, y) = x^{\alpha_1} y^{\beta_1} \cdots x^{\alpha_s} y^{\beta_s} = 1$$

where α_i, β_j are nonzero integers determined by w and $\alpha_1 < 0, \beta_s > 0$.

Proof. If $w = w(x_1, \ldots, x_r)$, then replacing x_i by $x^{-i} y x^i$, we get a nontrivial group identity in the two variables x and y satisfied by $U(R)$, namely

$$x^{\alpha_1} y^{\beta_1} \cdots x^{\alpha_s} y^{\beta_s} x^{\alpha_{s+1}} = 1$$

where α_i, β_j are nonzero integers with the exception of α_{s+1} which is possibly zero. Then $U(R)$ satisfies

$$x^{\alpha_1 + \alpha_{s+1}} y^{\beta_1} \cdots x^{\alpha_s} y^{\beta_s} = 1.$$

If $\alpha_1 + \alpha_{s+1} \neq 0$, we can get the desired form by replacing x by x^{-1} or y by y^{-1} if necessary. If $\alpha_1 + \alpha_{s+1} = 0$, then we get

$$y^{\beta_1} x^{\alpha_2} \cdots y^{\beta_{s-1}} x^{\alpha_s} y^{\beta_s} = 1.$$

By repeating the above process, we either get the desired form or $U(R)$ satisfies a group identity of the form $x^\alpha = 1$ for some $\alpha \neq 0$. Then $U(R)$ satisfies $(x^{-1}y)^{|\alpha|} = 1$ which is the form we want. □

Let R be a ring with identity. We say that R is locally Artinian if any finite subset X of R is contained in an Artinian subring of R. Certainly, if G is a locally finite group, then $K[G]$ is a locally Artinian ring.

The following lemma is in [Kar89, p.43].

Lemma 2.2. *Let R be a ring and suppose $R/J(R)$ is Artinian where $J(R)$ is the Jacobson radical of R. If I is any ideal of R, then the natural map $U(R) \to U(R/I)$ is surjective.*

With this, we can lift units in locally Artinian rings.

Lemma 2.3. *Let R be a locally Artinian ring and let $U(R)$ satisfy $w = 1$. If S is any subring of R or $\bar{R}$ is any homomorphic image of R, then $U(S)$ and $U(\bar{R})$ also satisfy $w = 1$.*

Proof. The result for $U(S)$ is obvious since $U(S) \subseteq U(R)$. For $\bar{R}$, it is sufficient to show that the homomorphism $U(R) \to U(\bar{R})$ is surjective. To this end, let a be any element in R with the image $\bar{a} \in U(\bar{R})$. Let $b \in R$ such that $\bar{b}$ is the inverse of $\bar{a}$. Then a and b are contained in an Artinian subring E of R since R is locally Artinian. Lemma 2.2 implies that $U(E) \to U(\bar{E})$ is surjective. But $\bar{a} \in U(\bar{E})$, hence we can find $c \in U(E) \subseteq U(R)$ such that c maps to $\bar{a}$. □

We use p to denote the characteristic of the field K. And for convenience we make the convention that when $p = 0$, any group is a $0'$-group and the only 0-group is $\{1\}$. Recall that a group A is called p-abelian if the commutator subgroup A' is a finite p-group.

We need some classical results about group algebras.

Lemma 2.4 (Isaacs-Passman)**.** *If $\text{char} K = p \geq 0$, then $K[G]$ satisfies a polynomial identity if and only if G contains a p-abelian subgroup of finite index.*

Proof. See [Pas85, p.196-197]. □

Let

$$\Delta(G) = \{g \in G | g \text{ has only finitely many conjugates in } G\},$$
$$\Delta^p(G) = \langle g \in \Delta(G) | g \text{ is a } p\text{-element}\rangle$$

Let $N = N(K[G])$ be the nilpotent radical of $K[G]$, namely, the sum of all nilpotent ideals of $K[G]$.

Lemma 2.5. *Suppose $\text{char} K = 0$; then $K[G]$ is semiprime.*

Proof. See [Pas85, p.130]. □

Lemma 2.6 (Passman)**.** *Suppose $\text{char} K = p > 0$. The following are equivalent:*

1. *$K[G]$ is semiprime.*
2. *G has no finite normal subgroup with order divisible by p.*
3. *$\Delta(G)$ has no element of order p.*
4. *$N(K[G]) = 0$.*

Proof. See [Pas85, p.131 and p.309]. □

Lemma 2.7 (Passman)**.** *Suppose $\text{char} K = p > 0$. Then $N(K[G])$ is nilpotent if and only if $\Delta^p(G)$ is finite.*

Proof. See [Pas85, p.311]. □

Let R be a K-algebra. We say that g is a multilinear generalized polynomial of degree n if

$$g(x_1, x_2, \ldots, x_n) = \sum_{\sigma \in Sym_n} g^\sigma(x_1, x_2, \ldots, x_n)$$

and

$$g^\sigma(x_1,\dots,x_n)=\sum_{j=1}^{a_\sigma}\alpha_{0,\sigma,j}x_{\sigma(1)}\alpha_{1,\sigma,j}x_{\sigma(2)}\cdots\alpha_{n-1,\sigma,j}x_{\sigma(n)}\alpha_{n,\sigma,j}$$

where $\alpha_{i,\sigma,j}\in R$ and a_σ is some positive number. We say that g is nondegenerate if for some $\sigma\in Sym_n$, g^σ is not a generalized polynomial identity for R. R is said to satisfy a GPI if R satisfies a nondegenerate multilinear generalized polynomial identity. We have a well-known lemma:

Lemma 2.8. *Let R be a K-algebra and let I be a right ideal of R. If I satisfies a polynomial identity of degree k and $I^k\neq 0$, then R satisfies a GPI.*

Proof. By a standard linearization process, I satisfies a multilinear polynomial identity of degree k:

$$g(x_1,\dots,x_k)=\sum_{\sigma\in Sym_k}\alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(k)}$$

where $\alpha_\sigma\in K$ and $\alpha_1\neq 0$. Since $I^k\neq 0$, we can choose $a_1,\dots,a_k\in I$ such that $a_1a_2\cdots a_k\neq 0$. Then $\alpha_1a_1a_2\cdots a_k\neq 0$ implies that

$$\sum_{\sigma\in Sym_k}\alpha_\sigma a_{\sigma(1)}x_{\sigma(1)}a_{\sigma(2)}x_{\sigma(2)}\cdots a_{\sigma(k)}x_{\sigma(k)}$$

is a nondegenerate multilinear generalized polynomial identity for R. □

The following result is from [Pas85, p.202].

Lemma 2.9 (Passman)**.** *$K[G]$ satisfies a GPI if and only if $[G:\Delta(G)]<\infty$ and $|\Delta(G)'|<\infty$.*

3. Group algebras satisfying a GPI

Our goal in this section is to prove that if $U(K[G])$ satisfies a group identity and $K[G]$ satisfies a GPI, then $K[G]$ satisfies a polynomial identity.

We use K_0 to denote the integers in K. First we prove an analogue of [GJV94, Proposition 1], using a variant of their argument.

Lemma 3.1. *Let R be an algebra over a field K and suppose $U(R)$ satisfies $w=1$. Then there exists a polynomial $f(x)$ over K_0 of degree d which is determined by the word w such that if $a,b\in R$ and $a^2=b^2=0$, then $f(ab)=0$.*

Proof. By Lemma 2.1, we may assume $U(R)$ satisfies

$$w_0(x_1,x_2)={x_1}^{\alpha_1}{x_2}^{\beta_1}\cdots{x_1}^{\alpha_s}{x_2}^{\beta_s}=1$$

where α_i,β_j are nonzero integers and $\alpha_1<0,\beta_s>0$.

Replacing x_1 by $x_2x_1^{-1}$ and x_2 by $x_1^{-1}x_2$, the group identity $w_0=1$ becomes

$$w_1(x_1,x_2)={x_1}^{\gamma_1}{x_2}^{\delta_1}\cdots{x_1}^{\gamma_k}{x_2}^{\delta_k}=1$$

where $\gamma_i,\delta_j\in\{\pm1,\pm2\}$ and $\gamma_1=\delta_k=1$.

If $char K\neq 2$, consider the polynomial h in two variables given by

$$\begin{aligned}h(x_1,x_2)&=(1+\gamma_1x_1)(1+\delta_1x_2)\cdots(1+\gamma_kx_1)(1+\delta_kx_2)-1\\&=h_0(x_1,x_2)+g_{11}(x_1,x_2)+g_{12}(x_1,x_2)+g_{21}(x_1,x_2)+g_{22}(x_1,x_2)\end{aligned}$$

where h_0 consists of all monomials which contain x_1^2 or x_2^2, and where g_{ij} contains all remaining monomials which start with x_i and end with x_j. Notice that each monomial in $g_{12}(x_1,x_2)$ has the form $x_1x_2\cdots x_1x_2=(x_1x_2)^n$ for some n, and the leading term of $h(x_1,x_2)$ is in $g_{12}(x_1,x_2)$ with coefficient $\gamma_1\delta_1\cdots\gamma_k\delta_k\neq 0$ since $charK\neq 2$. Hence we can write $x_2g_{12}(x_1,x_2)x_1=f(x_2x_1)$ for some polynomial $f(x)$ over K_0 of degree $d=k+1$. Certainly, $f(x)$ is determined by $h(x_1,x_2)$, hence by w. Now $a^2=b^2=0$ implies that $1+a$ and $1+b$ are units of R. Also, $(1+a)^n=1+na$ for any integer n. Thus $w(1+b,1+a)=1$ gives us

$$\begin{aligned}0&=a(w_1(1+b,1+a)-1)b=ah(b,a)b\\&=a(h_0(b,a)+g_{11}(b,a)+g_{12}(b,a)+g_{21}(b,a)+g_{22}(b,a))b.\end{aligned}$$

Now $ah_0(b,a)b=0$ since each term of h_0 has a^2 or b^2 as a factor. Also

$$a(g_{21}(b,a)+g_{22}(b,a))b=0$$

since each term of g_{21} or g_{22} starts with a, and $ag_{11}(b,a)b=0$ since each term of g_{11} ends with b. This gives us $f(ab)=ag_{12}(b,a)b=0$.

If $charK=2$, we replace x_1 by x_1x_2 and x_2 by x_1x_3, so the group identity $w_1=1$ becomes

$$w_2(x_1,x_2,x_3)=(x_1x_2)^{\gamma_1}(x_1x_3)^{\delta_1}\cdots(x_1x_2)^{\gamma_k}(x_1x_3)^{\delta_k}=1$$

where $\gamma_1=\delta_k=1$. Writing it out, we get a reduced form

$$w_3(x_1,x_2,x_3)=z_1^{\eta_1}z_2^{\eta_2}\cdots z_r^{\eta_r}$$

where $z_i\in\{x_1,x_2,x_3\}$, $z_i\neq z_{i+1}$ and $\eta_i\in\{\pm1\}$. Furthermore, $z_1=x_1$, $z_r=x_3$ and $\eta_1=\eta_r=1$.

Define a polynomial h of two variables by

$$h(x_1,x_2)=(1+t_1)(1+t_2)\cdots(1+t_r)-1$$

where

$$t_i=\begin{cases}x_1x_2x_1 & \text{if } z_i=x_1,\\ x_2x_1x_2x_1x_2 & \text{if } z_i=x_2,\\ (x_1+x_2x_1)x_2(x_1+x_1x_2) & \text{if } z_i=x_3.\end{cases}$$

In particular, $t_1=x_1x_2x_1$ and $t_r=(x_1+x_2x_1)x_2(x_1+x_1x_2)$. We now write $h=h_0+g_{11}+g_{12}+g_{21}+g_{22}$ as the previous case. By looking at the monomial $t_1t_2\cdots t_r$ in h, we see that $g_{12}\neq 0$, and hence $x_2g_{12}(x_1,x_2)x_1=f(x_2x_1)$ for some nonzero polynomial $f(x)\in K_0[x]$. Of course, $f(x)$ is determined by $h(x_1,x_2)$, and hence by w. Now, because $(bab)^2=(ababa)^2=((b+ab)a(b+ba))^2=0$, it follows that $1+bab, 1+ababa, 1+(b+ab)a(b+ba)$ are units. Note that in $charK=2$, $(1+c)^{-1}=1-c=1+c$ for any c such that $c^2=0$. Then

$$w_3(1+bab,1+ababa,1+(b+ab)a(b+ba))=1$$

yields

$$\begin{aligned}0&=a(w_3(1+bab,1+ababa,1+(b+ab)a(b+ba))-1)b=ah(b,a)b\\&=a(h_0(b,a)+g_{11}(b,a)+g_{12}(b,a)+g_{21}(b,a)+g_{22}(b,a))b.\end{aligned}$$

Arguing as in the previous case, we conclude that $f(ab)=ag_{12}(b,a)b=0$. □

For the rest of the paper we fix the notation so that $f(x)$ and d are as in Lemma 3.1.

Lemma 3.2. *Let R be an algebra over a field K and suppose $U(R)$ satisfies $w = 1$. Let $a, b \in R$ such that $a^2 = b^2 = 0$.*

1. *If $|K| > d$; then $(ab)^d = 0$.*
2. *If ab is nilpotent, then $(ab)^d = 0$.*

Proof. First we suppose $|K| > d$. For any $\lambda \in K$, $(\lambda a)^2 = b^2 = 0$. By Lemma 3.1, there exists $f(x) \in K[x]$ of degree d such that $f(\lambda ab) = 0$. Write $f(x) = \sum_{i=0}^{d} a_i x^i$ where $a_d \neq 0$. Then we have $a_d(ab)^d\lambda^d + \cdots + a_1 ab\lambda + a_0 = 0$ for any $\lambda \in K$. If $|K| > d$, a routine Vandermonde determinant argument shows that $a_d(ab)^d = 0$. Hence $(ab)^d = 0$.

Now we suppose that ab is nilpotent. Lemma 3.1 shows that there exists $f(x) \in K[x]$ of degree d such that $f(ab) = 0$. Let $g(x)$ be the minimal polynomial of ab over K. Then $g(x) = x^k$ for some k since ab is nilpotent. Also $g(x)|f(x)$ implies $k \leq d$. Thus $(ab)^d = 0$ since $0 = g(ab) = (ab)^k$. □

Remark. The first part of the above lemma is a consequence of [GJV94, Proposition 1] since the Vandermonde argument applies when the field is big enough. The second part is not used in this paper, but we still list it here because it works for algebras over arbitrary fields and will be used later.

Consider $M_n(F)$, the n by n matrix algebra over a field F. If $U(M_n(F))$ satisfies $w = 1$ and $n \geq 2$, then we get control of both the size of the field F and the degree n. Indeed, we have the following:

Lemma 3.3. *Let F be any field. If $U(M_n(F))$ satisfies $w = 1$ and $n \geq 2$, then*

1. *$|F| \leq d$ and hence F is a finite field.*
2. *$n < 2\log_{|F|} d + 2 \leq 2\log_2 d + 2$.*

Proof. Let $a = e_{12}$ and $b = e_{21}$, so $a^2 = b^2 = 0$. Since $e_{11} = ab$ is not nilpotent, it follows from Lemma 3.2 that $|F| \leq d$. Now let s be the smallest positive integer such that $|F|^s > d$; then $|F|^{s-1} \leq d < |F|^s$. Let E be the finite field such that $[E : F] = s$, so that $|E| = |F|^s > d$. Since E acts on E by right multiplication, we can embed E into $End_F(E) \cong M_s(F)$ and hence $M_{2s}(F) = M_2(M_s(F)) \supseteq M_2(E)$. If $n \geq 2s$, then $U(M_{2s}(F))$ satisfies $w = 1$, and hence $U(M_2(E))$ satisfies $w = 1$. But now $M_2(E)$ is an algebra over E with $|E| > d$, and this contradicts part 1. Therefore $n < 2s$ and $s - 1 \leq \log_{|F|} d$ yields the result. □

The following result is from [Gon84, Lemma 2.0].

Lemma 3.4. *Let D be a noncommutative division ring finite dimensional over its center. Then $U(D)$ contains a free subgroup of rank two.*

Now we can get the result about locally finite p'-group algebras.

Lemma 3.5. *Let K be a field of characteristic $p \geq 0$ and let G be a locally finite p'-group. If $U(K[G])$ satisfies $w = 1$, then $K[G]$ satisfies the standard polynomial identity $s_{2m} = 0$ for some integer m determined by the word w.*

Proof. Let m be an integer larger than $2\log_2 d + 2$. Let $\alpha_1, .., \alpha_{2m}$ be any $2m$ elements in $K[G]$ and let H be the subgroup of G generated by all the supports of the α_i. Since G is locally finite, H is finite, and therefore $K[H]$ is Artinian. Also H contains no p-elements, so $K[H]$ is semiprimitive and we have $K[H] = \bigoplus \sum_{i=1}^{r} M_{n_i}(D_i)$ for some integers n_i and division K-algebras D_i. Certainly, each

$U(D_i)$ satisfies $w = 1$ and $[D_i : Z_i] \leq [D_i : K] < \infty$ where Z_i is the center of D_i. By Lemma 3.4, D_i must be commutative. Next, each $U(M_{n_i}(D_i))$ satisfies $w = 1$, so Lemma 3.3 implies that each $n_i \leq m$. Consequently, $K[H]$ satisfies $s_{2m} = 0$. But $\alpha_1, \ldots, \alpha_{2m} \in K[H]$, hence $s_{2m}(\alpha_1, \ldots, \alpha_{2m}) = 0$. Therefore, $K[G]$ satisfies $s_{2m} = 0$. □

Lemma 3.6. *Let G be a torsion group. If $U(K[G])$ satisfies $w = 1$ and $K[G]$ satisfies a GPI, then $K[G]$ satisfies a polynomial identity.*

Proof. Let us write $\Delta = \Delta(G)$. By Lemma 2.9, $[G : \Delta] < \infty$ and $|\Delta'| < \infty$. Also, G torsion now implies that G is locally finite. Let C be the centralizer of Δ' in Δ; then $[\Delta : C] < \infty$ and C' is contained in the center of C. Let P be the set of all p-elements in C; then P is a normal subgroup of C since C is a nilpotent group. Also, since C is locally finite, $U(K[C/P])$ satisfies $w = 1$ by Lemma 2.3. Now C/P is a locally finite p'-group, so Lemma 3.5 implies $K[C/P]$ satisfies a polynomial identity. Hence C/P contains a p-abelian subgroup A/P of finite index by Lemma 2.4. C/P is a p'-group implies that A/P is abelian. Now, $[G : A] = [G : \Delta][\Delta : C][C : A] < \infty$ and $A' \subseteq P \cap \Delta'$ is a finite p-group. Hence Lemma 2.4 yields the result. □

4. Proof of the Main Theorem

We record some lemmas we need for semiprime case. The following is [GSV, Lemma 2.1].

Lemma 4.1. *Let R be a semiprime ring and let $S = \{a \in R : \text{for all } b, c \in R, bc = 0 \text{ implies } bac = 0\}$. If S contains all elements of R of square zero, then S contains all nilpotent elements of R.*

The following is a simplified version of [GJV94, Lemma 2].

Lemma 4.2. *Let R be an algebra over a field K. Suppose that for all $a, b, c \in R$ with $a^2 = bc = 0$, we have $bac = 0$. Then every idempotent of R is central.*

We note a simple consequence of Lemma 3.1.

Lemma 4.3. *Let R be an algebra over a field K and let $U(R)$ satisfy $w = 1$. If $a, b, c \in R$ such that $a^2 = bc = 0$, then all elements of $bacR$ satisfy some polynomial $g(x) \in K_0[x]$ of degree $d + 1$.*

Proof. For any $r \in R$, $a^2 = (crb)^2 = 0$. By Lemma 3.1, there exists an $f(x) \in K_0[x]$ of degree d such that $f(acrb) = 0$. Let $g(x) = xf(x)$. Then $g(bacr) = bacrf(bacr) = bf(acrb)acr = 0$. □

We are now ready to prove the semiprime case (compare with [GJV94, Theorem 6] and [GSV, case(i)]).

Theorem 4.4. *Let K be a field of char$K = p \geq 0$ and let G be a torsion group. Suppose $K[G]$ is a semiprime group algebra. If $U(K[G])$ satisfies $w = 1$, then $K[G]$ satisfies a polynomial identity.*

Proof. Let us write $k = d + 1$. There are two cases to consider.

Case 1: There exist elements $a, b, c \in K[G]$ such that $a^2 = bc = 0$, and $(bacK[G])^k \neq 0$.

Case 2: For all elements $a, b, c \in K[G]$ such that $a^2 = bc = 0$, we have $(bacK[G])^k = 0$.

For case 1, Lemma 4.3 implies that $bacK[G]$ satisfies a polynomial identity of degree k. By Lemma 2.8, $K[G]$ satisfies a GPI since $(bacK[G])^k \neq 0$. Lemma 3.6 therefore yields the result for case 1.

Now, let us consider case 2. For all $a, b, c \in K[G]$ such that $a^2 = bc = 0$, we have $(bacK[G])^k = 0$. Hence $bac = 0$ by semiprimeness of $K[G]$. By Lemma 4.2, all idempotents of $K[G]$ are central. By Lemma 4.1, if $a, b, c \in K[G]$ such that a is nilpotent and $bc = 0$, then $bac = 0$.

Let P be the set of all p-elements and let Q be the set of all p'-elements in G. If $p = 0$, $P = \{1\}$. Suppose $charK = p > 0$. Take any $h \in P$ and write $\widehat{h} = 1 + h + h^2 + \cdots + h^{o(h)-1}$. For any $g \in P$, $g - 1$ is nilpotent and $(h-1)\widehat{h} = 0$. Hence we have $0 = (h-1)(g-1)\widehat{h} = (h-1)g\widehat{h}$. It follows that $hg\widehat{h} = g\widehat{h}$ and $g = hgh^i$ for some i. Thus $g^{-1}hg = h^{-i}$. This shows that P is a group and $\langle h \rangle \lhd P$.

For any $h \in P$, $g \in Q$, $h-1$ is nilpotent and $(g-1)\widehat{g} = 0$. Hence $(g-1)(h-1)\widehat{g} = 0$. Computing as above, we get $h = ghg^j$ for some j and $h^{-1}gh = g^{-j}$. Note that $g^{-1}h^{-1}gh = g^{-j-1} \in P \cap Q = 1$ since P is a normal subgroup of G. It follows that for any $h \in P$, $\langle h \rangle$ commutes with Q and is normalized by P. Thus $\langle h \rangle \lhd G$ and $h = 1$ by semiprimeness of $K[G]$. Therefore, for $charK = p > 0$, we also get $P = \{1\}$.

Now for $charK = p \geq 0$, take any $x \in Q$ with order m. Since $m \neq 0$ in K, $e = \widehat{x}/m$ is an idempotent. Since all idempotents are central, we see that for any $g \in G$, $xg = gx^i$ for some i. This shows that Q is a group and $\langle x \rangle$ is normal in G. Therefore, Q is abelian or Hamiltonian. If Q is Hamiltonian, then it contains $\mathbf{Q}_8$, the quaternion group of order 8. Note that Q is a p'-group, and consequently so is $\mathbf{Q}_8$. Therefore, $K[\mathbf{Q}_8]$ is semiprimitive and Artinian. We can write $K[\mathbf{Q}_8] = \bigoplus \sum M_{n_i}(D_i)$. In $K[G]$, all idempotents are central, and hence the same holds in $K[\mathbf{Q}_8]$. It follows that each $n_i = 1$. Also, by Lemma 3.4, each D_i is commutative. Then $K[\mathbf{Q}_8]$ is commutative, a contradiction. Hence Q is abelian. Since G is generated by P and Q, we conclude that G is abelian and hence $K[G]$ satisfies s_2. This settles case 2. □

Remark. In case 2 above, the argument still holds if we only suppose that G is torsion-generated.

Lemma 4.5. *Let K be a field of characteristic $p > 0$ and let G be a group. If $N(K[G])$ is nilpotent, then G has subgroups P and H such that P is a finite p-group, $[G : H] < \infty$, and $P = \Delta^p(H)$.*

Proof. See [Pas85, p.312-313]. □

Theorem 4.6. *Let K be a field of characteristic $p > 0$ and let G be a torsion group. If $N(K[G])$ is nilpotent and $U(K[G])$ satisfies $w = 1$, then $K[G]$ satisfies a polynomial identity.*

Proof. Since $N(K[G])$ is nilpotent, we can choose subgroups P and H of G as in Lemma 4.5. Now G is torsion and $\Delta^p(H) = P$ is a finite p-group, so $\Delta(H/P) = \Delta(H)/\Delta^p(H)$ is a p'-group, and hence $K[H/P]$ is semiprime. Note that the natural map $U(K[H]) \to U(K[H/P])$ is onto since P is a finite p-group. Hence $U(K[H/P])$ also satisfies $w = 1$. Then Theorem 4.4 implies that $K[H/P]$ satisfies a polynomial identity. Hence H/P has a p-abelian subgroup A/P of finite index by Lemma 2.4. Note that A' is a finite p-group, and $[G : H] < \infty$. Therefore A is a p-abelian subgroup of G of finite index. This implies that $K[G]$ satisfies a polynomial identity by Lemma 2.4. □

The following proof modifies the argument given in [GSV, case(iii)].

Theorem 4.7. *Let K be a field of characteristic $p > 0$ and let G be a torsion group. If $N(K[G])$ is not nilpotent and $U(K[G])$ satisfies $w = 1$, then $K[G]$ satisfies a polynomial identity.*

Proof. Let t be an indeterminate and let $K\{X\}[[t]]$ denote the power series ring over the free algebra $K\{X\}$ where $X = \{x_1, x_2, \ldots\}$. For any n, the elements $1 + x_1 t, 1 + x_2 t, \ldots, 1 + x_n t$ are units in $K\{X\}[[t]]$ and they generate a free group by the well-known result of Magnus. In particular, if w is a word in k variables, then $w(1 + x_1 t, \ldots, 1 + x_k t) \neq 1$. It follows that we get an expression of the form

$$w(1 + x_1 t, \ldots, 1 + x_k t) - 1 = \sum_{i \geq 1} f_i(x_1, \ldots, x_k) t^i \neq 0,$$

where $f_i(x_1, \ldots, x_k) \in K\{X\}$ is a homogeneous polynomial of degree i. Thus there exists a smallest integer $s \geq 1$ such that $f_s(x_1, \ldots, x_k) \neq 0$, and we can write

$$w(1 + x_1 t, \ldots, 1 + x_k t) - 1 = \sum_{i \geq s} f_i(x_1, \ldots, x_k) t^i \neq 0.$$

Now we want to show that for any integer n, there exists a nilpotent ideal I of $K[G]$ such that $I^n \neq 0$. Suppose by way of contradiction that $I^n = 0$ for any nilpotent ideal I of $K[G]$. Let $a_1, \ldots, a_n \in N(K[G])$. There exist finitely many nilpotent ideals $I_1, \ldots, I_j$ such that $\{a_1, \ldots, a_n\} \subseteq J = \sum_{i=1}^{j} I_i$. J is nilpotent, hence $J^n = 0$. It follows that $a_1 a_2 \cdots a_n = 0$ and $(N(K[G]))^n = 0$, a contradiction. Hence we can find a nilpotent ideal I of $K[G]$ such that $I^r \neq 0$ and $I^{r+1} = 0$ for some integer $r > s$.

For any $a_1, \ldots, a_k, a_{k+1} \in I$, $1 + a_1, \ldots, 1 + a_k$ are units of $K[G]$. Thus we have

$$0 = w(1 + a_1, \ldots, 1 + a_k) - 1 = \sum_{i=s}^{r} f_i(a_1, \ldots, a_k)$$

since $I^{r+1} = 0$. Then

$$0 = (w(1 + a_1, \ldots, 1 + a_k) - 1) a_{k+1}^{r-s} = \sum_{i=s}^{r} f_i(a_1, \ldots, a_k) a_{k+1}^{r-s}$$

For $i \geq s + 1$, $f_i(a_1, \ldots, a_k) a_{k+1}^{r-s} = 0$ since $f_i(x_1, \ldots, x_k) x_{k+1}^{r-s}$ is homogeneous of degree $i + r - s \geq r + 1$ and $I^{r+1} = 0$. Consequently, $f_s(a_1, \ldots, a_k) a_{k+1}^{r-s} = 0$. Therefore, $f_s(x_1, \ldots, x_k) x_{k+1}^{r-s}$ is a polynomial identity of degree r for I. Moreover, $I^r \neq 0$. Thus, by Lemma 2.8, we conclude that $K[G]$ satisfies a GPI and hence a polynomial identity by Lemma 3.6. □

Remark. Actually, by the argument of [YC96, Fact 3], we can find a nilpotent ideal I such that $I^s \neq 0$ and $I^{s+1} = 0$. In this case, $f_s(x_1, \ldots, x_k)$ is a polynomial identity of degree s for I.

Theorems 4.4, 4.6 and 4.7 combine to yield Theorem 1.1.

Acknowledgment

The material in this paper is part of the author's Ph.D. thesis written at the University of Wisconsin. The author would like to thank his advisor, Professor Donald S. Passman, for helpful suggestions and continuous encouragement.

References

[BRT] Y. Billig, D. Riley, and V. Tasić, *Non-matrix varieties and nil-generated algebras whose units satisfy a group identity*, J. Algebra **190** (1997), no. 1, 241–252. CMP 97:10

[GJV94] A. Giambruno, E. Jespers, and A. Valenti, *Group identities on units of rings*, Arch. Math. (Basel) **63** (1994), no. 4, 291–296. MR **95h:**16044

[GM91] J. Z. Gonçalves and A. Mandel, *Semigroup identities on units of group algebras*, Arch. Math. (Basel) **57** (1991), no. 6, 539–545. MR **93g:**16049

[Gon84] J. Z. Gonçalves, *Free subgroups of units in group rings*, Canad. Math. Bull. **27** (1984), no. 3, 309–312. MR **85k:**20021

[GSV] A. Giambruno, S. Sehgal, and A. Valenti, *Group algebras whose units satisfy a group identity*, Proc. Amer. Math. Soc. **125** (1997), 629–634. MR **97f:**16056

[Kar89] G. Karpilovsky, *Unit groups of group rings*, Longman Scientific & Technical, 1989. MR **91h:**16001

[Pas] D. S. Passman, *Group algebras whose units satisfy a group identity II*, Proc. Amer. Math. Soc. **125** (1997), no. 3, 657–662. MR **98c:**16033

[Pas85] D. S. Passman, *The algebraic structure of group rings*, Robert E. Krieger, 1985. MR **86j:**16001

[YC96] Chi-Tsuen Yeh and Chen-Lian Chuang, *Nil polynomials of prime rings*, J. Algebra **186** (1996), no. 3, 781–792. MR **97k:**16031

Department of Mathematics, University of Wisconsin-Madison, Madison, Wisconsin 53706

E-mail address: chliu@math.wisc.edu