A BOUND FOR $|G : O_p(G)|_p$ IN TERMS OF THE LARGEST IRREDUCIBLE CHARACTER DEGREE OF A FINITE $p$-SOLVABLE GROUP $G$

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Abstract. Let $b(G)$ denote the largest irreducible character degree of a finite group $G$, and let $p$ be a prime. Two results are obtained. First, we show that, if $G$ is a $p$-solvable group and if $b(G) < p^2$, then $p^2 
ot| |G : O_p(G)|$. Next, we restrict attention to solvable groups and show that, if $b(G) \leq p^\alpha$ and if $P$ is a Sylow $p$-subgroup of $G$, then $|P : O_p(G)| \leq p^{2\alpha}$.

1. Introduction

Suppose $G$ is a finite group. Let $cd(G)$ denote the set $\{ \chi(1) \mid \chi \in \text{Irr}(G) \}$, and let $b(G)$ denote the largest irreducible character degree of a group $G$. Theorem 12.29 of [1] states: Let $p$ be a prime and let $b(G) < p$. Then $G$ has a normal abelian Sylow-$p$ subgroup. It is immediate from this result that if $b(G) < p$, then $p \nmid |G : O_p(G)|$. Later, in the same chapter of [1], we have Theorem 12.32: Suppose $b(G) < p^{3/2}$ for some prime $p$. Then $p^2 \nmid |G : O_p(G)|$. These facts raise the question: if $b(G) < p^\alpha$ for a real number $\alpha$, what can be said about the $p$ part of $|G : O_p(G)|$?

We address this question in the case when $G$ is a $p$-solvable group and obtain the following two results:

**Theorem A.** Let $G$ be a $p$-solvable group and let $p$ be a prime. If $b(G) < p^2$, then $p^2 \nmid |G : O_p(G)|$.

**Theorem B.** Let $G$ be a solvable group, let $p$ be a prime, and let $\alpha$ be a real number. If $b(G) \leq p^\alpha$ and if $P$ is a Sylow $p$-subgroup of $G$, then $|P : O_p(G)| \leq p^{2\alpha}$. In addition, if $|G|$ is odd, then $|P : O_p(G)| \leq p^\alpha$.

2. Preliminaries

In this section, we establish several facts regarding coprime actions that will be needed in the proofs of Theorem A and Theorem B.

**Theorem (2.1) (Brodkey).** Let $G$ be a finite group and assume that $S \in \text{Syl}_p(G)$ is abelian. Then there exists $T \in \text{Syl}_p(G)$ with $S \cap T = O_p(G)$.

*Proof.* This is Theorem 5.28 of [2].

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Lemma (2.2). Suppose that a $p$-group $P$ acts on a $p'$-group $H$. If $P$ fixes every character in $\text{Irr}(H)$, then the action of $P$ on $H$ is trivial.

Proof. If $P$ fixes every character of $H$, then, by Brauer’s Theorem (6.32 of [1]), $P$ fixes every conjugacy class of $H$. Since the size of a conjugacy class of $H$ is a $p'$-number, it follows that each conjugacy class contains a fixed point of $P$. Let $C = C_H(P)$ be the subgroup of fixed points. Since $C$ meets each class of $H$ nontrivially, it follows that $H$ is the (setwise) union of $H$-conjugates of $C$. This forces $H = C$, and thus the action of $P$ on $H$ is trivial. 

Last, we prove a special case of Theorem A.

Proposition (2.3). Suppose an abelian $p$-group $P$ acts faithfully on an abelian $p'$-group $H$, and let $G = H \rtimes P$. If $b(G) < p^2$, then $|P| \leq p$.

Proof. If $x \in C_P(h)$, then $h = h^x$; thus $x = x^h$ and, since $x \in P$, it follows that $x \in P \cap P^h$. Thus, in the action of $P$ on $H$, the stabilizer of a point $h$ in $H$, which is $C_P(h)$, is contained in $P \cap P^h$. Since the action of $P$ on $H$ is faithful and $H$ is abelian, Brauer’s Theorem (6.32 of [1]) implies that the action of $P$ on the abelian group $\text{Irr}(H)$ is faithful. Since $P$ is abelian, Brodkey’s Theorem (2.1) together with the fact that point stabilizers are contained in Sylow intersections implies that there is a regular orbit of $P$ on $\text{Irr}(H)$. Thus, for some character $\lambda \in \text{Irr}(H)$, we have $I_G(\lambda) = H$. It follows that $\lambda^G \in \text{Irr}(G)$, and therefore $|P| = |G : H| = \lambda^G(1) \leq b(G)$. Since $b(G) < p^2$, the conclusion holds. 

3. Proof of Theorem A

In this section we prove Theorem A. We begin with a technical lemma.

Lemma (3.1). Suppose that $G$ is a group with subgroups $H, P, A$, and $B$ such that $G = HP$ and $P = AB$. If $B \leq N_G([H, A])$, then $[H, A] \triangleleft G$ and $[H, A] \cdot [H, B] = [H, P]$.

Proof. Assume that $B \leq N_G([H, A])$. Since $H$ and $A$ normalize $[H, A]$, we have that $[H, A] \triangleleft G$ and it follows that the product $[H, A] \cdot [H, B]$ is a subgroup.

Clearly $[H, A] \cdot [H, B] \leq [H, P]$. We will show that this is an equality. Let $[h, x]$ be a generator of $[H, P]$, with $h \in H$ and $x \in P$. Write $x = ab$, where $a \in A$ and $b \in B$. One can check that:

$$[h, x] = [h, ab] = [h, b][h, a] = [H, a] \cdot [H, B] = [H, A] \cdot [H, B].$$

It follows that $[H, A] \cdot [H, B] = [H, P]$. 

Proof of Theorem A. Let $G$ be a counterexample of minimal order. We may assume that $O_p(G) = 1$. Set $H = O_{p'}(G)$. By the famous Lemma 1.2.3 of Hall and Higgins, see Lemma 14.22 of [1], we have $C_G(H) \leq H$.

First, we will prove a fact that will be used repeatedly: If $K$ is a subgroup of $G$ with $H \leq K$, then $O_p(K) = 1$. Assume that $K \geq H$. Since $H$ is a $p'$-group, $H$ and $O_p(K)$ are disjoint normal subgroups of $K$; thus $O_p(K) \leq C_K(H) \leq H$. Since $H$ has $p'$ order, this forces $O_p(K) = 1$.

Now fix $P \in \text{Syl}_p(G)$. As we have seen, $O_p(HP) = 1$. Since $P \in \text{Syl}_p(HP)$ and since $G$ is a minimal counterexample, we have $G = HP$.

Next, we will show that $P$ is abelian of order $p^2$. Let $P_1 \leq P$ with $|P : P_1| = p$. Since $H \leq HP_1$, we have $O_p(HP_1) = 1$. Since $|HP_1| < |HP|$, the minimality of $G$ implies that $|P_1| \leq p$. It follows that $|P| = p^2$ and that $P$ is abelian.
Let $\psi \in \text{Irr}(H)$ and $\chi \in \text{Irr}(G|\psi)$. By Clifford’s Theorem (6.1 of [1]), we have $\chi|_H = e \sum_{i=1}^{t} \psi_i$, where $\{\psi_i\}_{i=1}^{t}$ is the complete orbit of $\psi$ in the conjugation action of $G$ on $\text{Irr}(H)$, labeled so that $\psi_1 = \psi$. Also $et$ divides $|G : H| = |P| = p^2$, by Corollary 11.29 of [1]. Since $\chi(1) = et\psi(1)$ and since $et$ divides $p^2$, the hypothesis on character degrees of $G$ implies that $et \leq p$. It follows that $et$ divides $p$. We claim that $e = 1$. Suppose, for a contradiction, that $e > 1$. Then $e = p$ and $t = 1$; consequently, $\psi$ is $G$-invariant. Since $H$ is a normal Hall subgroup of $G$, it follows that $\psi$ extends to $G$ (see Gallagher’s Theorem 8.15 of [1]). Further, since $P$ is abelian, every irreducible character of $G$ that lies over $\psi$ must be an extension (Corollary 6.17 of [1]); however, this contradicts $\chi_H = p\psi$. Thus, as claimed, $e = 1$, and it follows that $\chi|_H = \sum_{i=1}^{p} \psi_i$ or $\chi_H = \psi$.

Now let $A$ be a subgroup of $P$ that fixes every character in $\text{Irr}(H)$. By Lemma (2.2), $A \leq C_P(H) \leq H$, and, since $A$ is a $p$-group, this implies that $A = 1$. Thus, the action of $P$ on $\text{Irr}(H)$ is faithful. Next we will deduce that $P$ must be an elementary abelian $p$-group. Let $I_P(\psi)$ denote the stabilizer in $P$ of $\psi$. As we have seen, for every character $\psi \in \text{Irr}(H)$, either $|P : I_P(\psi)| = 1$ or $p$; as a consequence, $|I_P(\psi)| > 1$. If $P$ is cyclic, then $P$ has a unique subgroup of order $p$. This subgroup would have to be contained in $I_P(\psi)$, for every character $\psi \in \text{Irr}(H)$. This contradicts the fact that the action on $\text{Irr}(H)$ is faithful. Thus $P$ is not cyclic, and, therefore, is elementary abelian of order $p^2$.

Next we will show that $H = [H, P]$. By properties of coprime actions, we have that $H = [H, P] \cdot C_H(P)$. If $X \leq P$, then $[H, X] \triangleleft G$, since $H$ normalizes $[H, X]$ and $P$ normalizes both $H$ and $X$. Also, if $X$ is nontrivial, then $[H, X]$ is nontrivial, since $C_G(H) \leq H$. In particular, $[H, P]$ is a nontrivial normal subgroup of $G$. Now consider $[H, P] \cdot P$. Observe that $O_p([H, P] \cdot P)$ and $[H, P]$ are disjoint normal subgroups of $[H, P] \cdot P$, therefore they centralize each other. Of course, $O_p([H, P] \cdot P)$ is contained in $P$ and centralizes $C_H(P)$, thus it follows that $O_p([H, P] \cdot P)$ centralizes $H = [H, P] \cdot C_H(P)$. Since the action of $P$ on $H$ is faithful, we have $O_p([H, P] \cdot P) = 1$. Further, $P \in \text{Syl}_p([H, P] \cdot P)$; therefore, by the minimality of $G$, we have $G = [H, P] \cdot P$. Thus, we may conclude that $H = [H, P]$.

Let $M \leq H$ be a minimal normal subgroup of $G$; we will show that $M = [H, X]$ for some nontrivial subgroup $X \leq P$. Consider the group $G/M$. Since $G$ is a minimal counterexample and since a Sylow $p$-subgroup of $G/M$ is isomorphic to $P$, we must have $O_p(G/M) > 1$. Let $X \leq P$ such that $O_p(G/M) = XM/M$. Then $X > 1$ and $[H, X] \leq M$. As we have seen, $1 < [H, X] \triangleleft G$ for every nontrivial subgroup $X \leq P$; thus $M = [H, X]$.

We now make several observations about an arbitrary subgroup $A$ of $P$, with $|A| = p$. We have seen that $A$ must move some character in $\text{Irr}(H)$. Also, we know that $1 < [H, A] \triangleleft G$.

Assume now that $[H, A] < H$, and let $B = O_p(P \cdot [H, A])$. We will show that $A$ and $B$ have the following dual relationship:

(i) $|H, B| < H$ and $A = O_p(P \cdot [H, B])$;
(ii) $B = C_P([H, A])$ and $A = C_P([H, B])$;
(iii) $|B| = |A| = p$ and $A \cap B = 1$, thus $P = AB$.

By properties of coprime actions, $H = [H, A] \cdot C_H(A)$. Now consider the group $P \cdot [H, A]$. This is a proper subgroup of $G$, since $[H, A]$ is proper in $H$; also, $P \in \text{Syl}_p(P \cdot [H, A])$. Since $G$ is a minimal counterexample, it follows that $B > 1$. Next observe that $[H, A]$ and $B$ are disjoint normal subgroups of $P \cdot [H, A]$,
hence $B \leq C_P([H, A])$. Using properties of coprime actions again, we have that $[[H, A], A] = [H, A]$, which is nontrivial. It follows that $A$ does not centralize $[H, A]$, however, $B$ does, and therefore $A \nsubseteq B$. Since $B$ is nontrivial and $|P| = p^2$, it follows that $|B| = p$. Also, since $A$ is nontrivial, we have that $P = AB$ and $A \cap B = 1$; thus statement (iii) is proved. Further, since $A \nsubseteq C_P([H, A])$, we have $C_P([H, A]) < P$. Thus $1 < B \leq C_P([H, A]) < P$, and it follows that $B = C_P([H, A])$. Thus the first statement in (ii) is proved.

Observe that, since $B$ centralizes $[H, A]$ and since $P$ is abelian, we have $[[A, H], B] = 1 = [[B, A], H]$. By the Three Subgroups Theorem, it follows that $[[H, B], A] = 1$, and thus $A \leq C_P([H, B])$. Since $A$ is not centralized by $H$, we have that $[H, B] < H$, and thus the first statement in (i) holds. Further, since $C_P([H, B])$ is contained in the abelian group $P$, we have $C_P([H, B]) \leq O_p(P \cdot [H, B])$, and, since $[H, B] < H$, the same reasoning that we used to show that $O_p(P \cdot [H, A])$ is proper in $P$ yields that $O_p(P \cdot [H, B])$ is proper in $P$. Since $A$ is nontrivial, it follows that $A = C_P([H, B]) = O_p(P \cdot [H, B])$, and the rest of the assertion has been proved.

When the situation arises that $[H, A] < H$, we will call the group $B$ thus identified the dual of $A$.

Now suppose that $A$ is a subgroup of $P$ of order $p$, and assume that $[H, A]$ is proper in $H$. We claim that $[H, A]$ is a minimal normal subgroup of $G$. Since $1 < [H, A] \triangleleft G$, we may fix a minimal normal subgroup $M$ of $G$ with $M \leq [H, A]$. As we have seen, $M = [H, X]$ for some nontrivial subgroup $X \leq P$. If $X = A$, then $M = [H, A]$ and the claim holds. Otherwise, $X \neq A$, in which case $P = XA$, and Lemma (3.1) yields

$$H = [H, X] \cdot [H, A] \leq M \cdot [H, A] \leq [H, A].$$

This contradicts the fact that $[H, A]$ is proper in $H$. Therefore $X = A$, and hence $[H, A]$ is minimal normal.

Continue to assume that $A$ is a subgroup of $P$ of order $p$ with $[H, A]$ proper in $H$, and let $B$ be the dual of $A$. We will show that $H = [H, A] \times [H, B]$, where $[H, A]$ and $[H, B]$ are minimal normal subgroups of $G$. Since $[H, A]$ is proper in $H$, the duality of $A$ and $B$ implies that $[H, B]$ is proper in $H$, and thus both $[H, A]$ and $[H, B]$ are minimal normal subgroups of $G$. Since $P = AB$, Lemma (3.1) yields $[H, A] \cdot [H, B] = H$. If $[H, A] \cap [H, B] > 1$, then, by the minimality of the factors, we have $[H, A] = [H, B]$, and hence $H = [H, A]$, which is a contradiction. It follows that $[H, A] \cap [H, B] = 1$, and thus $H = [H, A] \times [H, B]$.

We will now show that, in fact, there are no subgroups $A \leq P$ of order $p$ with $[H, A]$ proper in $H$. Assume, for a contradiction, that such a subgroup $A$ does exist, and let $B$ be the dual of $A$. Then $P = A \times B$, and $H = [H, A] \times [H, B]$. We claim that

$$G = ([H, A] \cdot A) \times ([H, B] \cdot B).$$

Each of $[H, A] \cdot A$ and $[H, B] \cdot B$ is a subgroup of $G$. By duality, $A$ centralizes $[H, B]$, and thus $A$ centralizes $[H, B] \cdot B$. Also $[H, A]$ centralizes $[H, B]$, since these are disjoint normal subgroups, and $[H, A]$ centralizes $B$, by the duality of $A$ and $B$. Thus $[H, A] \cdot A$ centralizes $[H, B] \cdot B$. Since $H = [H, A] \cdot [H, B]$ and $P = AB$, we have that $G = ([H, A] \cdot A) \cdot ([H, B] \cdot B)$. To see that this is a direct product, we consider the orders of the factors. Since $H$ and $P$ are direct products, we have
that $|H| = |[H, A]| \cdot |[H, B]|$ and $|P| = |A| \cdot |B|$. Since $G = HP$, it follows that

$$|G| = |H| \cdot |P| = |A| \cdot |[H, A]| \cdot |B| \cdot |[H, B]|.$$ 

Finally, since $G = ([H, A] \cdot A) \cdot ([H, B] \cdot B)$, we can deduce that $([H, A] \cdot A) \cap ([H, B] \cdot B) = 1$, and thus $G = ([H, A] \cdot A) \times ([H, B] \cdot B)$.

Next, observe that $A$ must move some character in Irr($[H, A]$). If not, then by Lemma (2.2), $A$ acts trivially on $[H, A]$. However, by properties of coprime actions, $[[H, A], A] = [H, A]$, which contradicts the fact that $[H, A] > 1$. Thus, the direct factor $[H, A] \cdot A$ has some irreducible character of degree divisible by $p$. The same is true for $[H, B] \cdot B$, hence $G$ must have an irreducible character of degree less than $p^2$ and this contradicts the hypothesis on $b(G)$. Therefore, for every subgroup $A$ of order $p$, we must have $[H, A] = H$.

For our last observation before returning to character theory, we will show that $H$ is the direct product of isomorphic nonabelian simple groups. First, we will show that $H$ is a minimal normal subgroup of $G$. Let $M$ be minimal normal in $G$ with $M \leq H$. Then $M = [H, X]$ for a nontrivial subgroup $X \leq P$. Since $X > 1$, we have $H = [H, X]$, and therefore $H = M$. It now follows that $H$ is the direct product of isomorphic simple groups. Further, if one of these direct factors of $H$ is abelian, then $H$ is abelian and Proposition (2.3) leads to a contradiction. It follows that $H$ has the claimed structure.

Let $\psi \in \text{Irr}(H)$, and assume that $\psi$ is moved by $P$. Let $A = I_P(\psi)$, and let $\chi \in \text{Irr}(G|\psi)$. We have seen that $1 < A < P$, and that $\chi_H$ is the sum of $p$ distinct conjugates of $\psi$. Since $\chi(1) < p^2$, it follows that $\psi(1) \leq p - 1$. Now, let $C = C_H(A)$; notice that $H = [H, A] \cdot C$ and that $P \leq N_G(C)$, since $P$ centralizes $A$ which uniquely determines $C$. Also note that $\ker(\psi) < H$, since the trivial character is, of course, invariant.

We now consider $\psi_C$. By Theorem 13.14 of [1], which follows from the Glauberman correspondence, we have that $\psi_C = a\alpha + p\Phi$ where $\alpha \in \text{Irr}(C)$, $a \equiv \pm 1 \pmod{p}$, and $\Phi$ is a, possibly zero, character of $C$. Since $\psi(1) \leq p - 1$, we have $\Phi = 0$, and $a = 1$ or $a = p - 1$. Thus, either $\psi_C = \alpha$ or $\psi_C = (p - 1)\alpha$, where $\alpha \in \text{Irr}(C)$, and if the latter case holds, then $\alpha$ is linear. The character $\alpha$ is the Glauberman correspondent of $\psi$. Note that, since $\psi$ is nontrivial, $\alpha$ is nontrivial as well. Next we will see that, in fact, the case $\psi_C = \alpha$ never occurs.

Suppose that $\psi_C = \alpha$. Let $\hat{\psi}$ be the canonical extension of $\psi$ to the inertial subgroup $I = I_G(\psi)$; thus $\hat{\psi}$ is the unique extension of $\psi$ to $I$ with $\alpha(\hat{\psi})$ a $p'$-number. The existence of $\hat{\psi}$ is guaranteed by Gallagher’s Theorem (Corollary 6.28 of [1]). Let $R$ be an irreducible representation that affords $\hat{\psi}$. Then $[R(C), R(A)] = 1$, since $C$ is centralized by $A$. Also, $R_C$ is irreducible, since it affords the irreducible character $\alpha$. By Schur’s Lemma, we have that $R_A$ is a scalar representation. Thus $[R(H), R(A)] = 1$, and it follows that $[H, A] \leq \ker(\psi)$. Further, since $[H, A] \leq H$, we have $[H, A] \leq \ker(\psi) \cap H = \ker(\psi) < H$. However, we have shown that $[H, A] = H$ for every subgroup $A \leq P$ with $|A| = p$. This contradiction implies that, for every non-$P$-invariant character $\psi \in \text{Irr}(H)$, we have $\psi_C = (p - 1)\alpha$ for some nontrivial linear character $\alpha \in \text{Irr}(C)$.

For the final contradiction, we consider what is known about $\psi$. Since $\psi$ is not $P$-invariant, we have $\psi \neq 1_H$, and thus its Glauberman correspondent $\alpha$ is a nontrivial linear character of $C$. Since $\alpha$ is nontrivial, $C$ is not contained in $\ker(\psi)$, and since $\alpha$ is linear, $Z(\psi)$ is contained in $C$; thus $Z(\psi) > \ker(\psi)$. Since each
of \( Z(\psi) \) and \( \ker(\psi) \) is normal in \( H \), it follows that \( H \) has a nontrivial abelian (in fact, cyclic) section \( Z(\psi)/\ker(\psi) \). This contradicts the fact that \( H \) is the direct product of nonabelian simple groups. Thus, our minimal counterexample \( G \) cannot exist.

### 3. Proof of Theorem B

In this section, we consider what can be said if the character degree hypothesis of Theorem A is weakened to \( b(G) \leq p^\alpha \). If attention is restricted to solvable groups, then, as a direct consequence of a theorem of D. Passman, Theorem B is gained.

**Proof of Theorem B.** Without loss of generality, we may assume that \( 1 = O_p(G) < G \). For \( P \in \text{Syl}_p(G) \), our aim is to show that \( |P| \leq p^{2\alpha} \). Set \( H = O_{p'}(G) \), and note that \( H \) is nontrivial, since \( G \) is solvable with \( O_p(G) = 1 \). As in the proof of Theorem A, we may assume that \( G = HP \).

Next we show that we may assume that \( H \) is nilpotent, or equivalently, that \( H = F(G) \). Since \( O_p(G) = 1 \), we have \( F(G) \leq H \). Further, since \( G \) is solvable, \( C_G(F(G)) \leq F(G) \). It follows that \( O_p(F(G) \cdot P) = 1 \), and, since \( b(F(G) \cdot P) \leq b(G) \leq p^\alpha \), the hypotheses hold in the group \( F(G) \cdot P \). Since \( P \in \text{Syl}_p(F(G) \cdot P) \), we may hence assume that \( H \) is nilpotent.

Now, we consider the coprime action of \( P \) on the abelian group \( \text{Irr}(H/\Phi(H)) \). Since the action of \( P \) on the nilpotent group \( H \) is coprime and faithful, it follows that the action of \( P \) on the abelian group \( \text{Irr}(H/\Phi(H)) \) is faithful. By Corollary 2.4 of [3], there exists \( \lambda \in \text{Irr}(H/\Phi(H)) \), such that the \( P \)-orbit of \( \lambda \) has size at least \( \sqrt{|P|} \), and thus for any character \( \chi \in \text{Irr}(G|\lambda) \), we have \( \chi(1) \geq \sqrt{|P|} \). Since \( p^\alpha \geq b(G) \geq \chi(1) \), it follows that \( p^\alpha \geq \sqrt{|P|} \), and therefore \( |P| \leq p^{2\alpha} \).

Finally, we observe that, if \( |G| \) is odd, then Corollary 2.4 of [3] asserts that there exists \( \lambda \in \text{Irr}(H/\Phi(H)) \), such that the \( P \)-orbit of \( \lambda \) has size at least \( |P| \). In this case, it follows that \( |P| \leq p^\alpha \).

### References


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