TOEPLITZ $C^*$-ALGEBRAS ON ORDERED GROUPS
AND THEIR IDEALS OF FINITE ELEMENTS

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Abstract. Let $G$ be a discrete abelian group and $(G, G_+)$ an ordered group. Denote by $(G, G_F)$ the minimal quasily ordered group containing $(G, G_+)$. In this paper, we show that the ideal of finite elements is exactly the kernel of the natural morphism between these two Toeplitz $C^*$-algebras. When $G$ is countable, we show that if the direct sum of $K$-groups $K_0(T^{G+}_G) \oplus K_1(T^{G+}_G) \cong Z$, then $K_0(T^{G+}_G) \cong Z$.

1. Preliminaries and the main result

Let $G$ be a discrete abelian group and $G_+$ a subset of $G$, $(G, G_+)$ is said to be a quasily ordered group, if $0 \in G_+ \subseteq G_+$ and $G = G_+ \cup (-G_+)$. If in addition, $G_0^0 = G_+ \cap (-G_+) = \{0\}$, then $(G, G_+)$ is referred to as an ordered group.

In this section, we always fix an ordered group $(G, G_+)$. Let $\hat{G}$ be the dual group of $G$, $\hat{G}$ is compact and it is connected since $G$ is torsion-free. When given the normal Haar measure, it is easy to show that $\{\varepsilon_x | x \in G\}$ is an orthonormal basis for $L^2(\hat{G})$, where $\varepsilon_x$ is defined by $\varepsilon_x(\gamma) = \gamma(x)$ for $\gamma \in \hat{G}$.

For any $E \subseteq G$, let $H^2(E)$ be the closed subspace of $L^2(\hat{G})$ generated by $\{\varepsilon_x | x \in E\}$; its projection is denoted by $p^E$. For any $\varphi \in C(\hat{G})$, define $T^E_\varphi$ on $H^2(E)$ by $T^E_\varphi(f) = p^E(\varphi f)$ for $f \in H^2(E)$. We say that $T^E_\varphi$ is a Toeplitz operator (relative to $E$) with symbol $\varphi$. The $C^*$-algebra generated by $\{T^E_\varphi | \varphi \in C(\hat{G})\}$ is denoted by $\mathcal{T}^E$, and is called the Toeplitz $C^*$-algebra with respect to $E$. By the Stone-Weierstrass theorem, $\mathcal{T}^E$ is also generated by $\{T^E_{\varepsilon_x} | x \in G\}$.

Now let $(G, G_+)$ be an ordered group, denote by $F(G) = F(G_+) \cup (-F(G_+))$, where

$$F(G_+) = \{x \in G_+ | \forall y \in G_+ \setminus \{0\}, \exists n \in N, \text{ such that } ny - x \in G_+\}.$$ 

It is easy to show that $F(G)$ is a subgroup of $G$ and its elements are usually called the finite elements. Denote by $K(F(G))$, the closed two-sided ideal of $T^G_r$.
generated by
\[ \{ 1 - T_{g}^{G_{+}} T_{g}^{G_{+}} \mid x \in F(G_{+}) \}. \]
Let \( G_{F} = G_{+} \cup F(G) \), then obviously \( G_{F}^{G_{+}} = F(G) \), and it is easy to show that
\[ G_{F} = \bigcap_{g \in G_{+} \setminus \{0\}} (G_{+} - Z_{+} g) = G_{+} - F(G_{+}). \]

So when \( F(G) \neq \{0\} \), \((G,G_{F})\) is the minimal quasily ordered group containing \((G,G_{+})\).

A representation of an ordered group \((G,G_{+})\) by isometries on a Hilbert space \(H\), is a map \(V : G_{+} \to \mathcal{B}(H)\) such that
\[ V(0) = 1, V(x)^{\ast} V(x) = 1 \text{ and } V(x) V(y) = V(x + y) \text{ for all } x, y \in G_{+}. \]
The Toeplitz \(C^{\ast}\)-algebra \(T_{G_{+}}^{G_{+}}\) has a universal property for isometric representations, that is, if \((V,H)\) is any isometric representation of \(G_{+}\), then there is a unique \(C^{\ast}\)-morphism \(\pi_{V} : T_{G_{+}}^{G_{+}} \to \mathcal{B}(H)\), such that \(\pi_{V}(T_{g}^{G_{+}}) = V(x)\) for all \(x \in G_{+}\). Furthermore, if every \(V(x), x \neq 0\) is non-unitary, then \(\pi_{V}\) is isometric (see [1], Theorem 1.3 and Theorem 2.9 or [5], Section 5.2). So for any quasily ordered group \((G,E)\) with \(G_{+} \subseteq E\), there is a \(C^{\ast}\)-algebra morphism \(\gamma_{E,G_{+}}: T_{r}^{G_{+}} \to T_{r}^{E}\) such that
\[ \gamma_{E,G_{+}}(T_{g}^{G_{+}}) = T_{g}^{E} \text{ for all } g \in G_{+}. \]
Obviously, \(K(F(G)) \subseteq Ker \gamma_{G_{F},G_{+}}^{G_{+}}\). The main result of this paper is, we prove that \(K(F(G))\) is equal to \(Ker \gamma_{G_{F},G_{+}}^{G_{+}}\).

**Remark.** By an isometric representation \((V,H)\) of a quasily ordered group \((G,G_{F})\), we mean \(H\) is a Hilbert space and \(V : G_{F} \to \mathcal{B}(H)\) is a map satisfying
\[ V(0) = 1, V^{\ast}(x)V(x) = 1, V(x + y) = V(x)V(y) \text{ for all } x, y \in G_{F}; \]
\[ V(x)V^{\ast}(x) = 1 \text{ for all } x \in F(G). \]
The following lemma is crucial to our proof of Theorem 1.2, we defer its proof to Section 3.

**Lemma 1.1.** Any isometric representation \((V,H)\) of \(G_{F}\) on a Hilbert space \(H\) can be lifted to be a \(C^{\ast}\)-algebra morphism \(\pi_{V} : T_{r}^{G_{F}} \to \mathcal{B}(H)\), such that \(\pi_{V}(T_{x}^{G_{+}}) = V(x)\) for all \(x \in G_{F}\).

**Theorem 1.2.** Let \((G,G_{+})\) be an ordered group and \(F(G)\) the subgroup of finite elements, then \(K(F(G)) = Ker \gamma_{G_{F},G_{+}}^{G_{+}}\).

**Proof.** It needs only to prove \(“Ker \gamma_{G_{F},G_{+}}^{G_{+}} \subseteq K(F(G))”\). Let \(\rho\) be the natural morphism induced by \(\gamma_{G_{F},G_{+}}^{G_{+}}\) from \(T_{r}^{G_{+}}/K(F(G))\) to \(T_{r}^{G_{F}}\), then by Lemma 1.1 \(\rho\) has an inverse \(\pi_{V}\) lifted by the isometric representation of \(G_{F}\) on \(T_{r}^{G_{+}}/K(F(G))\) defined by
\[ V(x) = [T_{x}^{G_{+}}] \in T_{r}^{G_{+}}/K(F(G)) \text{ for } x \in G_{F}, \]
therefore, \(Ker \gamma_{G_{F},G_{+}}^{G_{+}} \subseteq K(F(G))\). □

Let \((G,G_{+})\) be as above and let \(V\) be a representation of \(G_{+}\) by isometries. If for every \(x \in G_{+}\) with \(x \neq 0\), \(V(x)\) is non-unitary, then \(T_{r}^{G_{+}} \cong C^{\ast}(\{V(x) \mid x \in G_{+}\})\). On the other hand, if there does exist some \(y \in G_{+}, y \neq 0\) such that \(V(y)\) is a
unitary, then it is easy to show that $V$ is also an isometric representation of $G_F$ (when $x \notin G_+$, we define $V(x) = V(-x)^*$). Indeed, for any $x \in F(G_+)$, by definition there is an $n \in \mathbb{N}$ such that $ny - x \in G_+$. So

$$V(ny - x)V(x)V(x)^* = V(ny)V(ny)^* = V(y)^nV(y)^* = 1.$$  

Multiply with $V(ny - x)^*$ on the left and $V(ny - x)$ on the right, we obtain $V(x)V(x)^* = 1$. Therefore, we have the following

**Corollary 1.3.** Let $(G, G_+)$ be an ordered group with $F(G) \neq \{0\}$. Then every non-faithful representation $\pi$ of $T^{G_+}_r \rightarrow \mathbb{B}(H)$ can be factored through $T^{G_F}_r$, in the sense that there exists a representation $\pi_F : T^{G_F}_r \rightarrow \mathbb{B}(H)$ such that $\pi = \pi_F \circ \gamma_{G_F,G_+}$.

**Remark.** (1) Let $(G, G_+)$ be an ordered group and $F(G)$ the subgroup of finite elements. By Corollary 1.3 and Theorem 1.2, it is easy to show that $K(F(G))$ is contained in any closed two-sided ideal of $T^{G_+}$. In particular, $K(F(G))$ is simple and $T^{G_+}_r$ is prime. Such a property was also established by G. Murphy in a far different way (see [1], Theorem 2.10 and Theorem 2.11). Our proof here in some sense is sensibly simpler than the original one.

(2) Suppose $F(G) \neq \{0\}$. If $G$ admits a least positive element, equivalently, $T^{G_+}_r$ contains a Fredholm operator of non-zero index, or $K(F(G))$ is the ideal of compact operators on $H^2(G_+)$ ([2], Lemma 2.2), then by Theorem 1.2 we know that

$$\forall T \in T^{G_+}_r, T \text{ is Fredholm if and only if } \gamma_{G_F,G_+}(T) \text{ is invertible in } T^{G_F}_r.$$  

When $(G, G_+) = (\mathbb{Z}, \mathbb{Z}_+)$, it reduces obviously to the classical case. A precise character of a single Toeplitz operator with continuous symbol to be Fredholm can be found in ([2], Theorem 2.5 and Theorem 4.2), and a generalized version of [2] has been given in [8].

2. A character of $K$-groups of Toeplitz $C^*$-algebras on ordered groups

Let $A$ be a $C^*$-algebra, denote by $K_0(A)$ and $K_1(A)$ the $K_0$-group and $K_1$-group of $A$ respectively. If $K_0(A) \oplus K_1(A) \cong \mathbb{Z}$, then obviously either $K_0(A) \cong \mathbb{Z}$, $K_1(A) = 0$ or $K_1(A) \cong \mathbb{Z}$, $K_0(A) = 0$. Let $T$ be the classical Toeplitz $C^*$-algebra, it is well-known that $K_0(T) \cong \mathbb{Z}$ and $K_1(T) = 0$. On the other hand, there are many $C^*$-algebras, such as $\mathbb{B}/\mathbb{K}$, $C_0(\mathbb{R}^{2n+1})$, their $K_1$-groups are all isomorphic to $\mathbb{Z}$ and their $K_0$-groups are all equal to zero. Quite surprisingly, in this section, we show that when $G$ is a countably infinite discrete abelian group and $(G, G_+)$ is an ordered group, if the direct sum of the $K$-groups $K_0(T^{G_+}_r) \oplus K_1(T^{G_+}_r)$ is isomorphic to $\mathbb{Z}$, then $K_0(T^{G_+}_r)$ must be isomorphic to $\mathbb{Z}$ (of course in this case $K_1(T^{G_+}_r)$ must be equal to zero).

In this section, we always assume $G$ is a countably infinite discrete abelian group. Let $(G, G_+)$ be an ordered group, a subgroup $I$ of $G$ is said to be an ideal, if $0 \leq x, y \leq \in I \cap G_+$ implies $x + y \in I$ for all $x \in G$, where “$\leq$” is the usual order on $G$ induced by $G_+$. For any $x \in G$, set $V_x = T^{G_+}_r \in T^{G_+}_r$. Let $I$ be an ideal of $G$, we define $T(G, I)$ to be the closed ideal in $T^{G_+}_r$ generated by $\{1 - V_x | x \in I \cap G_+ \}$. If $(G_1, (G_1)_+), \ldots , (G_n, (G_n)_+)$ are ordered groups, we denote by $(G_1 \times \cdots \times G_n, G_1 * \cdots * G_n)$ the lexicographic-ordered group.
Theorem 2.2. Let $(G_1, (G_1)_+)$ and $(G_2, (G_2)_+)$ be two ordered groups. Then there is a $C^*$-morphism $\alpha : T^G_1 \otimes G \rightarrow C(G_2) \otimes T^G_1$ such that for all $(x,y) \in G_1 \ast G_2$, we have $\alpha(V_{x,y}) = \varepsilon_y \otimes V_x$. Moreover $\text{Ker} \alpha = T(G_1 \times G_2, G_2)$.

Proof. We identify $\{0\} \times Z$ with $Z$. By the above lemma, we have the following short exact sequence of $C^*$-algebras:

$$0 \longrightarrow T(G \times Z, Z) \longrightarrow T^G \longrightarrow C(T) \otimes T^G \longrightarrow 0$$

where $i$ is the inclusion map and $\alpha(V_{x,n}) = \varepsilon_n \otimes V_x$ for any $(x,n) \in G \ast Z$. Since the subgroup of finite elements in $G \times Z$ is just $Z$, by ([2], Lemma 2.2) we know that $T(G \times Z, Z) = \mathbb{K}(H^2(G \ast Z))$, where $\mathbb{K}(H^2(G \ast Z))$ is the ideal of compact operators on $H^2(G \ast Z)$. So we have the following periodic six-term exact sequence of $K$-theory:

$$
\begin{array}{cccccc}
\xrightarrow{i_*} K_0(C(T) \otimes T^G) & \xrightarrow{\alpha_*} K_1(T^G) & \xrightarrow{\delta} K_1(K) = 0 \\
\xrightarrow{i_*} K_0(T^G) & \xrightarrow{\alpha_*} K_1(T^G) & \xrightarrow{i_*} K_1(K) = 0
\end{array}
$$

Since $C(T) \otimes T^G \cong C(T \rightarrow T^G)$, by ([7], Exercise 8.B) (or simply use the Kinneth theorem) we know that $K_0(C(T) \otimes T^G) \cong K_1(C(T) \otimes T^G) \cong K_0(T^G) \oplus K_1(T^G) \cong Z$. If we also denote the index map by $\delta$, then since $V_{(0,1)}$ is an isometry, by the definition of index map ([7], Definition 8.1.1) and ([7], Exercise 8.C) we know that $\delta[1 \otimes 1_{T^G}] = -[1 - V_{(0,1)} V^*_{(0,1)}] \in K_0(K)$, where $z \in C(T)$ is the inclusion of $T$ into $\mathbb{R}$. Note that any homomorphism from $Z$ to $Z$ is injective iff it is non-zero, and it is surjective iff 1 or $-1$ is in its range. Obviously, the rank of $1 - V_{(0,1)} V^*_{(0,1)}$ is 1, so the index map in the exact sequence is an isomorphism; therefore,

$$K_0(T^G) \cong Z, \quad K_1(T^G) = 0.$$ 

The map

$$G \ast Z \rightarrow T^G, \quad (x,n) \rightarrow V_x,$$

is an isometric homomorphism, and therefore induces a $C^*$-morphism $\rho : T^G \rightarrow T^G$ such that $\rho(V_{x,n}) = V_x$ for all $(x,n) \in G \ast Z$. Similarly, there is a $C^*$-morphism $\theta : T^G \rightarrow T^G$ such that $\theta(V_{x,n}) = V_{x,0}$ for all $x \in G_+$. Since $T^G$ is generated by $\{ V_x \mid x \in G_+ \}$, we know that $\rho \circ \theta = id_{T^G}$, so $\rho \circ \theta = id_{K_1(T^G)}$, it follows that the map $\rho_* : K_1(T^G) \rightarrow K_1(T^G) \cong \text{onto}$, so $K_1(T^G) = 0$. \qed

Remark. It is easy to show by induction that for any $n \in N$, $K_0(T^{Z_n \ast \ldots \ast Z}) \cong Z$ and $K_1(T^{Z_n \ast \ldots \ast Z}) = 0$, and this was established in [4]. Next we give another Toeplitz $C^*$-algebra, whose $K_0$-group is also isomorphic to $Z$ and whose $K_1$-group is trivial.

Example. Let

$$G = \mathbb{Z}^2, G_+ = \{ (m,n) \in \mathbb{Z}^2 \mid m+n > 0, m+n = 0 \text{ and } m \geq 0 \}.$$
Clearly, \((G, G_+)\) is an ordered group, \((1, -1)\) is the least positive element of \(G\) and 

\[
F(G) = \{ (n, -n) \mid n \in \mathbb{Z} \}.
\]

So 

\[
G_F = \{ (m, n) \in \mathbb{Z}^2 \mid m + n \geq 0 \}.
\]

By Theorem 1.2, we have the following short exact sequence of \(C^*\)-algebras:

\[
0 \longrightarrow K(H^2(G_+)) \longrightarrow \mathcal{T}_r^{G+} \longrightarrow \mathcal{T}_r^{G_F} \longrightarrow 0.
\]

Therefore, we have the following six-term exact sequence of \(K\)-theory:

\[
\begin{align*}
\mathbb{Z} \cong K_0(\mathbb{K}) & \xrightarrow{i} K_0(\mathcal{T}_r^{G+}) \xrightarrow{(\gamma^{G_F,G_+})_*} K_0(\mathcal{T}_r^{G_F}) \\
\xrightarrow{\text{index map}} & \frac{K_1(\mathcal{T}_r^{G+})}{(\gamma^{G_F,G_+})_*} \xrightarrow{i_*} K_1(\mathcal{K}) = 0
\end{align*}
\]

By ([6], Section 3 (set \(\alpha = -1\))), we know that \(T_r^{G_F} \cong C(T) \otimes \mathcal{T}\), where \(T\) is the classical Toeplitz \(C^*\)-algebra, so \(K_0(T_r^{G_F}) \cong \mathbb{Z} \cong K_1(T_r^{G_F})\). As before we know that the index map is an isomorphism, therefore we know that 

\[
K_0(T_r^{G_F}) \cong \mathbb{Z}, \quad K_1(T_r^{G_F}) = 0.
\]

3. The Proof of Lemma 1.1

The ideals of this section are mostly contained in [5]. Roughly speak, the quasi-lattice ordered groups discussed in [5] are “ordered groups”. What we consider here are the quasiordered groups; in other words, the unit space may be not trivial.

We deal with Toeplitz \(C^*\)-algebras in another context.

Let \(G\) be a discrete (not necessary to be abelian) group and \(P\) a subsemigroup of \(G\). Then \((G, P)\) is said to be a quasiordered group, if 

\[
e \in P, \ P \cdot P \subseteq P, \ \text{and} \ G = P \cup P^{-1},
\]

where \(e\) is the unit of \(G\) and \(P^{-1} = \{ x^{-1} \mid x \in P \}\).

Now let \((G, P)\) be a quasiordered group, denote by \(G_+^0 = P \cap P^{-1}\), it is a subgroup of \(G\). Since \((P \setminus G_+^0) \cdot P = (P \setminus G_+^0) = P \cdot (P \setminus G_+^0)\), if we let \(G_+ = (P \setminus G_+^0) \cup \{ e \}\), then \(G = G_+ \cup G_+^0 \cup (G_+)^{-1}\) and \((G, G_+)\) is a partially ordered group in the sense that 

\[
e \in G_+, \ G_+ \cdot G_+ \subseteq G_+, \ G_+ \cap G_+^{-1} = \{ e \}, \ \text{and} \ G = G_+ \cdot (G_+)^{-1}.
\]

For any \(x, y \in G\), we say 

\[
x \leq y \ (\text{resp.} \ x \ll y, \ x \sim y) \ \text{if} \ x^{-1}y \in G_+ \ (\text{resp.} \ x^{-1}y \in P, \ x^{-1}y \in G_+^0)\).
\]

Let \(\delta_g \mid g \in G\) be the usual orthonormal basis for \(\ell^2(G)\), where 

\[
\delta_g(h) = \begin{cases} 1, & \text{if} \ g = h, \\ 0, & \text{otherwise}. \end{cases} \text{for} \ g, h \in G.
\]

For any quasiordered group \((G, P)\), another Toeplitz \(C^*\)-algebra \(W(G, P)\) can be defined as the \(C^*\)-subalgebra of \(\mathcal{B}(\ell^2(P))\) generated by \(\{ J^*L_tJ \mid t \in G \}\) with \(L : G \to \mathcal{B}(\ell^2(G))\) the left regular representation and \(J : \ell^2(P) \to \ell^2(G)\) the inclusion operator. For any \(t \in P\), the operator \(J^*L_tJ\), which will now be denoted by \(w(t)\) and let \(w(t)w(t)^* = m(t)\) as in [5] for any \(t \in P\).
The formulas listed in Proposition 3.1 below are clear:

**Proposition 3.1.** (1) $w(s)w(t) = w(st), w(t)^*w(t) = 1, w(s) = w(t) \Leftrightarrow s = t, \forall s, t \in P$.

(2) $w(s_1)w(t_1)^* = w(s_2)w(t_2)^* \Leftrightarrow t_1 \sim t_2$ and $s_1t_1^{-1} = s_2t_2^{-1}$.

(3) $w(s)^*w(t) = \begin{cases} w(s^{-1}t), & \text{if } s \ll t, \\ w(t^{-1}s)^*, & \text{if } t \ll s. \end{cases}, \forall s, t \in P$.

(4) $m(s)m(t) = \begin{cases} m(s), & \text{if } t \ll s, \\ m(t), & \text{if } s \ll t. \end{cases}, m(s) = m(t) \Leftrightarrow s \sim t, \forall s, t \in P$.

**Proposition 3.2.** (1) Let $\mathcal{A} = \text{sp}\{w(s)w(t)^* | s, t \in P\}$, then $\mathcal{A}$ is a dense unital $*$-subalgebra of $W(G, P)$.

(2) The operators $\{w(s)w(t)^* | s, t \in P\}$ are linear independent in the sense that, if $\sum_j \lambda_j w(s_j)w(t_j)^* = 0$ with $w(s_j)w(t_j)^* \neq w(s_{j_1})w(t_{j_1})^*$ whenever $j_1 \neq j_2$, then $\lambda_j = 0$ for all $j$.

**Proof.** (1) It suffices to show for any $s_1, s_2, t_1, t_2 \in P, w(s_1)w(t_1)^*w(s_2)w(t_2)^* \in \mathcal{A}$, and this is clear by Proposition 3.1.

(2) If $T = \sum_j \lambda_j w(s_j)w(t_j)^* = 0$ with $w(s_j)w(t_j)^* \neq w(s_{j_1})w(t_{j_1})^*$ whenever $j_1 \neq j_2$, we show $\lambda_j = 0$ for all $j$. Without loss of generality, we may assume $t_1 \ll t_2 \ll \cdots \ll t_n$.

Suppose $t_1 \sim t_2 \sim \cdots \sim t_{k_1} < t_{k_1+1}$, then by $T \delta_{t_1} = 0$, we know that

$$\sum_{j=1}^{k_1} \lambda_j \delta_{s_jt_j^{-1}t_1} = 0.$$

If $\lambda_1 \neq 0$, then there exists $j \in \{2, 3, \ldots, k_1\}$ such that $s_jt_j^{-1}t_1 = s_1$, so $s_jt_j^{-1} = s_1t_1^{-1}$, which is a contradiction since $w(s_j)w(t_j)^* \neq w(s_1)w(t_1)^*$. Similarly, we have $t_2, t_3, \ldots, t_{k_1} = 0$. Continue the above process, eventually we have $\lambda_j = 0$ for all $j$. \qed

Denote by $D = \text{clos sp}\{m(t) | t \in P\}$. Clearly it is a unital abelian $C^*$-subalgebra of $W(G, P)$. Denote by $D_0$ the $C^*$-subalgebra of $B(\ell^2(P))$ consisting of all the operators having diagonal matrix relative to the canonical basis. It is well-known that there exists a linear and contractive map $E_0 : B(\ell^2(P)) \rightarrow D_0$ determined by the following rule: the matrix of $E_0(T)$ (relative to the canonical basis) is obtained from the one of $T$ by replacing with zero all the entries which are not situated on the principal diagonal.

**Proposition 3.3** (cf. [5], Section 3.3 and Section 3.6). (1) $D = \{T \in W(G, P) | T \text{ has diagonal matrix relative to the canonical basis of } \ell^2(P)\}$.

(2) Let $E = E_0|_{W(G, P)}$, then $E$ is a faithful bounded linear map from $W(G, P)$ to $D$ such that for any $s, t$ in $P$:

$$E(w(s)w(t)^*) = \begin{cases} w(s)w(t)^*, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

**Proposition 3.4.** Let $\{L(t) | t \in P\}$ be a family of non-zero projections of the unital $C^*$-algebra $\mathcal{B}$ with $L(e) = 1$. Then there exists a $C^*$-algebra morphism $\rho :$
For $L$ as above, $\rho$ is faithful if and only if $L(s) = L(t) \iff s \sim t$ for any $s, t \in P$.

**Proof.** "$\Rightarrow$" clearly follows from Proposition 3.1. To prove "$\Leftarrow$" it suffices to show that, for any $t_1, t_2, \ldots, t_n \in P$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in C$:

$$\| \sum_{j=1}^n \lambda_j L(t_j) \| \leq \| \sum_{j=1}^n \lambda_j m(t_j) \|.$$  

Without loss of generality, we may assume $t_1 \ll t_2 \ll \cdots \ll t_n$, more explicitly,

$$t_1 \sim \cdots \sim t_{k_1} < t_{k_1+1} \sim \cdots \sim t_{k_m-1} < \cdots < t_{k_m} = t_n.$$  

Let $s_i = \lambda_1 + \cdots + \lambda_{k_i}$ for $i = 1, \ldots, m$, and $L_{i,i+1} = L(t_{k_i}) - L(t_{k_{i+1}})$ for $i = 1, \ldots, m - 1$. Since $T = \sum_{j=1}^n \lambda_j m(t_j)$ has diagonal matrix,

$$\| \sum_{j=1}^n \lambda_j m(t_j) \| = \sup_{a \in P} |\langle T \delta_a, \delta_a \rangle| = \sup_{a \in P} |\sum_j \lambda_j \langle m(t_j) \delta_a, \delta_a \rangle|$$

$$= \max \{|s_1|, |s_2|, \ldots, |s_m|\}.$$  

On the other hand,

$$\sum_{j=1}^n \lambda_j L(t_j) = (\lambda_1 + \cdots + \lambda_{k_1}) L(t_{k_1}) + \cdots + (\lambda_{k_m-1} + \cdots + \lambda_{k_m}) L(t_{k_m})$$

$$= s_1 L_{1,2} + s_2 L_{2,3} + \cdots + s_m L_{m-1,m} + s_m L(t_{k_m}).$$  

So

$$\| \sum_{j=1}^n \lambda_j L(t_j) \| = \max_{1 \leq i \leq m-1} \{ |s_i| \| L(t_{k_i}) \neq L(t_{k_{i+1}}) \}, |s_m|\}.$$  

Thus we obtain

$$\| \sum_{j=1}^n \lambda_j L(t_j) \| \leq \| \sum_{j=1}^n \lambda_j m(t_j) \|.$$  

The proof above shows also that, for $L$ as above, $\rho$ is faithful if and only if $L(s) = L(t) \iff s \sim t$ for any $s, t \in P$. \hfill $\Box$

**Remark.** Let $V$ be a representation by isometries of $P$ on some Hilbert space $H$, i.e.

$$V(t)^* V(t) = 1, \forall t \in P; V(s)V(t) = V(st), \forall s, t \in P;$$

$$V(e) = 1; V(s)V(s)^* = 1, \forall s \in G^a_{+}.$$  

then $V$ is always covariant in the sense that the condition (2) stated in Proposition 3.4 holds with $L(t) = V(t) V(t)^*, \forall t \in P$, and if $w(s_1)w(t_1)^* = w(s_2)w(t_2)^*$ for any $s_1, s_2, t_1, t_2 \in P$, then $V(s_1) V(t_1)^* = V(s_2) V(t_2)^*$. In fact, by Proposition 3.1, we know that $t_1 \sim t_2$ and $s_1 t_1^{-1} = s_2 t_2^{-1}$. Let $a = a_1^{-1} t_2 \in G^a_{+}$, then

$$V(s_2) V(t_2)^* = V(s_1 a) V(t_1 a)^* = V(s_1) V(a)(V(t_1) V(a))^*$$

$$= V(s_1) V(a) V(a)^* V(t_1)^* = V(s_1) V(t_1)^*.$$
It follows by Proposition 3.2 that there is a unique linear operator \( \pi_V : \mathcal{A} \to \mathbb{B}(H) \) such that

\[
\pi_V (w(s)w(t)^*) = V(s)V(t)^* \quad \text{for all } s, t \in P,
\]

which directly shows that for all \( s_1, s_2, t_1, t_2 \in P \):

\[
\pi_V (w(s_1)w(t_1)^*w(s_2)w(t_2)^*) = \pi_V (w(s_1)w(t_1)^*)\pi_V (w(s_2)w(t_2)^*).
\]

So \( \pi_V \) is actually a \(*\)-representation.

For any \( T \in \mathcal{A} \), let \( T = \sum j \lambda_j w(s_j)w(t_j)^* \), define

\[
\|T\| = \sup \{ \|\pi(T)\| \mid \pi \text{ is a unital } \ast\text{-representation of } \mathcal{A} \}.
\]

\( \|T\| \) is finite and actually not greater than \( \sum_j |\lambda_j| \), because each \( w(s_j)w(t_j)^* \) is a partial isometry. On the other hand, the canonical identification \( \text{id} : \mathcal{A} \to W(G, P) \) gives an injective unital \(*\)-representation, hence \( \|T\| > 0 \) if \( T \neq 0 \). It follows immediately that \( \| \cdot \| \) is a \( C^*\)-norm on \( \mathcal{A} \).

**Definition.** The completion of \( \mathcal{A} \) with respect to \( \| \cdot \| \) will be denoted by \( C^*(G, P) \) and will be called the universal \( C^*\)-algebra of \( (G, P) \). \( (G, P) \) is said to be **amenable** if \( \pi_{id} : C^*(G, P) \to W(G, P) \) is one-to-one.

**Remark.** \( (G, P) \) is amenable if and only if, for every isometric representation \( V : P \to \mathbb{B}(H) \), there is a \( C^*\)-algebra morphism \( \pi_V : W(G, P) \to \mathbb{B}(H) \) such that \( \pi_V (w(t)) = V(t) \) for all \( t \in P \).

The proof of the following theorem is essentially the same as that of [5], the details can be found in ([5], Section 4.3, Section 4.4 and Section 4.5).

**Theorem 3.5.** If \( G \) is amenable, then \( (G, P) \) is amenable.

Now suppose \( G \) is a discrete abelian group and \( (G, G_+) \) is an ordered group. Let \( P = G_F \) and define a unitary \( U : \ell^2(G) \to L^2(\hat{G}) \) by \( U\delta_g = \varepsilon_g \) for \( g \in G \), then \( U^* T_{g}^{G_+} U = w(g) \) for all \( g \in G_+ \), so the two Toeplitz \( C^*\)-algebras \( T_r^{G_+} \) and \( W(G, G_+) \) are unitarily equivalent. Since in this case \( G \) is amenable, by Theorem 3.5 we know that the Lemma 1.1 stated in Section 1 now holds.

**References**

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