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A FACTORIZATION THEOREM FOR THE DERIVATIVE OF A FUNCTION IN H^p

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Abstract. We show that a function G is the derivative of a function f in the Hardy space H^p of the unit disk D for $0 < p < \infty$ if and only if $G = F\Phi'$ where $F \in H^p$ and $\Phi \in BMOA$. Here, F can be chosen to be non-vanishing, $||\Phi||_{BMOA} \le 1$, and $||F||_{H^p} \le C||f||_{H^p}$. As an application, we characterize positive measures μ on the unit disk such that the operator $L_\mu g(\zeta) = \int_D g(z)\frac{d\mu(z)}{(1-\zeta\bar{z})^2}$ is bounded from the tent space T^p_∞ to H^p, where $\frac{1}{2} < p < \infty$.

Let D be the unit disk in the complex plane and suppose that T is its boundary, the unit circle. For $\zeta \in T$, let $\Gamma(\zeta)$ be the non-tangential approach region

$$\Gamma(\zeta) = \{z \in D : |1 - z\bar{\zeta}| < (1 - |z|^2)\}.$$

Associated with the approach regions $\Gamma(\zeta), \zeta \in T$, are the maximal functions Nu and Au. Recall that for a function u defined on D

$$Nu(\zeta) = \sup_{z \in \Gamma(\zeta)} |u(z)|,$$

and

$$Au(\zeta) = (\int_{\Gamma(\zeta)} |u(z)|^2 \frac{da(z)}{(1-|z|)^2})^{\frac{1}{2}},$$

where da is area measure on D. For $0 < p < \infty$ define T^p_∞ to be the space of functions u continuous on D such that

$$||u||_{T^p_\infty} = (\int_T (Nu)^p d\theta)^{\frac{1}{p}} < \infty,$$

where $d\theta$ is arclength measure on T and let T^p_2 be the space of functions u defined on D such that

$$||u||_{T^p_2} = (\int_T (Au)^p d\theta)^{\frac{1}{p}} < \infty.$$

More generally, for $\alpha > 1$ there are maximal functions N_α and A_α associated with the approach regions

$$\Gamma_\alpha(\zeta) = \{z \in D : |1 - z\bar{\zeta}| < \frac{\alpha}{2}(1 - |z|^2)\}$$

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of aperture α. It is shown in [CMS] that the space T^p_∞ (resp. T^p_2) has an equivalent description in terms of N_α (resp. A_α.) We will also need the space T^∞_2 which is the space of functions v defined on D such that

$$||v||_{T^\infty_2} = (\sup_I \frac{1}{|I|}\int_{T(I)} |v(z)|^2 \frac{da(z)}{1-|z|})^{\frac{1}{2}} < \infty.$$

Here, I is an arc on the circle T, $T(I)$ is the "tent" over I (see [CMS]) and $|I|$ is the length of I. It is well known that the dual of T^1_2 is T^∞_2 with the pairing

$$\langle u, v\rangle = \int_D u(z)v(z)\frac{da(z)}{1-|z|}. \tag{1}$$

A holomorphic function f defined on D belongs to the Hardy space H^p if

$$||f||_{H^p} = (\int_T (Nf)^p d\theta)^{\frac{1}{p}} < \infty.$$

That is, for f holomorphic in D and $0 < p < \infty$, $||f||_{H^p} = ||f||_{T^p_\infty}$. The norm $||\ ||_{H^\infty}$ is the usual supremum norm. An alternative characterization of the Hardy spaces H^p, for $0 < p < \infty$ may be given in terms of T^p_2. Suppose that for $\beta > 0$ and f holomorphic on D, R^β is the fractional derivative of f of order β:

$$R^\beta f(z) = \sum_{n=0}^{\infty}(1+n)^\beta a_n z^n,$$

where $f(z) = \sum a_n z^n$ is the Taylor series of f. Let $\rho(z) = 1 - |z|$. It is well known that we have the equivalence of norms

$$||f||_{H^p} \doteq |f(0)| + ||\rho f'||_{T^p_2}, \tag{2}$$

where the notation $A \doteq B$ means that there is an absolute constant C such that $C^{-1}A \le B \le CA$. (We follow the practice of using the letter C to denote various numerical constants whose exact value will depend on the context in which it occurs.) Thus, $f \in H^p$ if and only if $\rho R^1 f \in T^p_2$. More generally (see [AB]), if $\beta > 0$ then a holomorphic function f defined on D belongs to H^p for $0 < p < \infty$ if and only if $A(\rho^\beta R^\beta f) \in L^p(d\theta)$. This statement is false if $\beta = 0$: the constant function $f = 1$ shows that a function in H^p need not belong to T^p_2 (in fact the only such function is the zero function). On the other hand, in [C1] the problem of determining the pointwise multipiers of H^p into T^p_2 was investigated. It was shown ([C1], Lemma 1) that if $h \in T^\infty_2$ then h has the property that $hf \in T^p_2$ for all functions $f \in H^p$. In fact the argument used in that proof establishes the following inequality.

Theorem A. *Let $0 < p < \infty$. There is a constant C depending only on p such that*

$$||uh||_{T^p_2} \le C||u||_{T^p_\infty}||h||_{T^\infty_2},$$

for all $u \in T^p_2$ and $h \in T^\infty_2$.

A type of converse to this result was given in [C2]. See also [CMS], pg. 320, and Remark 1 below.

Recall that one of the many characterizations of the class $BMOA$ of holomorphic functions on D which have bounded mean oscillation is in terms of Carleson measures and the space T_2^∞. A holomorphic function $\Phi \in BMOA$ if and only if the function $\rho\Phi' \in T_2^\infty$. Furthermore,

$$||\Phi||_{BMOA} = |\Phi(0)| + ||\rho\Phi'||_{T_2^\infty}$$

defines a norm equivalent to the intrinsic norm which is given in terms of $\Phi(0)$ and mean oscillation on arcs. See [G]. We therefore have the following fact.

Let $\Phi \in BMOA$. Then there is a constant $C(p)$ such that

$$(3) \qquad ||\rho f \Phi'||_{T_2^p} \leq C(p)||f||_{T_\infty^p}||\Phi||_{BMOA}.$$

The characterization of H^p in terms of T_2^p given by (2) combined with (3) yields the following corollary.

Corollary B. *Let $\Phi \in BMOA$, $F \in H^p$, and $0 < p < \infty$. Then $F\Phi'$ is the derivative of an H^p function f with $||f||_{H^p} \leq C||F||_{H^p}||\Phi||_{BMOA}$.*

In this note we prove the converse of this statement.

Theorem 1. *Let $0 < p < \infty$. A function G holomorphic in D is of the form $G = f'$ for $f \in H^p$ if and only if $G = F\Phi'$ where $F \in H^p$ and $\Phi \in BMOA$. Furthermore, F and Φ may be chosen so F is non-vanishing and outer in D, $\Phi(0) = 0$, $||\Phi||_{BMOA} \leq 1$ and $||F||_{H^p} \leq C(p)||f||_{H^p}$.*

Remark 1. In [CMS], pg. 320, it was observed that for $p > 2$, a function $u \in T_2^p$ if and only if $u = u_1 u_2$ where $u \in T_\infty^p$ and $u_2 \in T_2^\infty$. Thus Theorem 1 may regarded as a holomorphic version of this factorization.

Remark 2. For $0 < p, q < \infty$ let $F^{p,q}$ be the space of holomorphic functions G defined on D such that

$$||G||_{F^{p,q}} = (\int_0^{2\pi} (\int_0^1 |G(re^{i\theta})|^q (1-r)^{q-1} dr)^{\frac{p}{q}} d\theta)^{\frac{1}{p}} < \infty.$$

If $q = 2$, then $G \in F^{p,q}$ if and only if $G = f'$ for some $f \in H^p$. Since given $f \in H^p$ we may write $f' = F\Phi'$ with F non-vanishing, it follows easily that the zero sets of functions in the classes $F^{p,2}$, $0 < p < \infty$, coincide with the zero sets of derivatives of functions in the class $BMOA$. This result is in contrast to the result of Horowitz [H], Theorem 1, that given $p < q$ there is a zero set for the Bergman space A^p which is not a zero set for the Bergman space A^q. (See [H] for the relevant definitions.) It would be interesting to find a similar factorization theorem and results about zeros sets for the spaces $F^{p,q}$, $q \neq 2$.

As a second application of Theorem 1 we study the operator

$$(4) \qquad L_\mu f(\zeta) = \int_D \frac{f(z)}{(1-\zeta\bar{z})^2} d\mu(z),$$

where μ is a positive Borel measure defined on D. With the aid of Theorem 1 (and an appeal to a result of Luecking [Lu], Theorem 1) we characterize all measures μ with the property that L_μ defines a bounded operator from T_∞^p into H^p if $\frac{1}{2} < p < \infty$. For $z \in D$ and $0 < t < 1$ let $D_t(z) = \{w \in D : |z-w| < t(1-|z|)\}$. For a measure μ defined on D let $H^t\mu$ be the function defined by

$$H^t\mu(z) = \frac{1}{(1-|z|)^2} \int_{D_t(z)} d\mu.$$

Theorem 2. *Let $\frac{1}{2} < p < \infty$ and μ a positive Borel measure defined on D. Then the operator L_μ is bounded from T^p_∞ into H^p if and only if $H^t\mu \in T^\infty_2$ for some $0 < t < 1$.*

The proof of Theorem 1 is based on the following simple observation.

Lemma 3. *Suppose F is a holomorphic on D with positive real part. Then there is a constant C independent of F such that $||\rho \frac{F'}{F}||_{T^\infty_2} \leq C$. Furthermore, let $\log F$ be a logarithm of F with the property that $|Im \log F(0)| < \frac{\pi}{2}$. Then $\log F \in BMOA$ and there is a constant C independent of F such that $|| \log F||_{BMOA} \leq C$.*

Proof. Since $ReF(z) > 0$, $|Im \log F(z)| < \frac{\pi}{2}$ for all $z \in D$. The result follows since the Hilbert transform is a bounded operator from L^∞ into BMO. See [G], chapter VI, Theorem 1.5. □

Proof of Theorem 1. We now give the proof of Theorem 1. We need only show that if $G = f'$ where $f \in H^p$ then $G = F\Phi'$ where F and Φ are as in the statement of the theorem. We consider first the case where $p > 1$. Let $f \in H^p$. Then $|f| \in L^p(d\theta)$. Since $1 < p < \infty$, by the Marcel Riesz theorem (see [G], chapter 2, Theorem 2.3) we may find a holomorphic function F whose real part is the Poisson integral of $|f|$ such that $||F||_{H^p} \leq C||f||_{H^p}$. Here, C depends only on p. Note that $ReF(z) \geq |f(z)|$ for all $z \in D$. Write

$$f = F\frac{f}{F} = F\psi,$$

where $\psi = \frac{f}{F}$. Then $\psi \in H^\infty$, $||\psi||_\infty \leq 1$, and

$$\begin{aligned} f' &= F'\psi + F\psi' \\ &= F(\psi' + \psi\frac{F'}{F}) \\ &= F(\psi' + \psi(\log F)'). \end{aligned}$$

Let Φ be defined by the two conditions $\Phi' = \psi' + \psi(\log F)'$ and $\Phi(0) = 0$. Then the conclusion of the theorem (for the case $1 < p < \infty$) follows from Lemma 3.

We next consider the case where $0 < p < 2$. Let $G = f'$ where $f \in H^p$ and factor f as $f = BH^{\frac{2}{p}}$ where B is an inner function and H is an outer function with $H \in H^2$ with $||H||^2_{H^2} = ||f||^p_{H^p}$. As before, we may find F so $ReF \geq |H|$ and $||F||_{H^2} \leq C||H||_{H^2}$, where C depends only on p. Therefore

$$\begin{aligned} f &= B(\frac{H}{F})^{\frac{2}{p}} F^{\frac{2}{p}} \\ &= \psi F^{\frac{2}{p}}, \end{aligned}$$

where $||\psi||_\infty \leq 1$ and $||F^{\frac{2}{p}}||_{H^p} \leq C||f||_{H^p}$. Thus

$$\begin{aligned} f' &= \psi' F^{\frac{2}{p}} + \frac{2}{p} F^{\frac{2}{p}-1} F'\psi \\ &= F^{\frac{2}{p}}(\psi' + \frac{2}{p}\frac{F'}{F}\psi) \\ &= F^{\frac{2}{p}}(\psi' + \frac{2}{p}(\log F)'\psi) \end{aligned}$$

and the proof may be completed as in the first case. □

Proof of Theorem 2. We now give the proof of Theorem 2. For $z \in D$ and $0 < t < 1$ recall that $D_t(z) = \{w \in D : |z - w| < t(1 - |z|)\}$ and if μ is a positive measure defined on D, then

$$H^t\mu(z) = \frac{1}{(1-|z|)^2} \int_{D_t(z)} d\mu.$$

For a function u defined on D let $M^t u(z) = \sup_{w \in D_t(z)} |u(w)|$.

Suppose that $H^t\mu \in T_2^\infty$ for some $t > 0$. We show that if $1/2 < p < \infty$, then L_μ is a bounded operator from T_∞^p to H^p.

Consider first the case where $p > 1$. By duality, it suffices to show there is a constant C independent of g or f such that

$$|\int_T \overline{(L_\mu g)(\zeta)} f(\zeta) d\zeta| \leq C ||g||_{T_\infty^p} ||f||_{H^{p'}}, \tag{5}$$

where p' is the exponent conjugate to p. Interchanging the order of integration in the integral in (5) shows that it is enough to prove that

$$\int_D |g(z)f'(z)| d\mu(z) \leq C ||g||_{T_\infty^p} ||f||_{H^{p'}}.$$

Write $f' = F\Phi'$ where F and Φ are as in Theorem 1. Since the spaces T_∞^p are independent of the aperture of the approach region, it is easily verified that for $t < 1$ there is a constant depending on t but not g or f such that

$$||M^t(gF)||_{T_\infty^1} \leq C ||g||_{T_\infty^p} ||f||_{H^{p'}}.$$

Furthermore, since $|\Phi'|^2$ is subharmonic, there is also a constant C depending only on t such that

$$||\rho M^t(\Phi')||_{T_2^\infty} \leq C.$$

Let $G(z) = |g(z)f'(z)| = |g(z)F(z)\Phi'(z)|$. It follows from Theorem A that for any $t < 1$, there is a constant C depending only on t and p such that

$$||\rho M^t G||_{T_2^1} \leq C ||g||_{T_\infty^p} ||f||_{H^p}.$$

For a function u defined on D and $0 < t < 1$ let $H^t u = H^t(u da)$. It is easily seen that we may choose s, independent of G, $0 < s < t$, so

$$G(z) \leq C H^s(M^t G)(z)$$

for all $z \in D$. It follows from Fubini's theorem and the duality given by the pairing (1) that

$$\begin{aligned}
\int_D |g(z)f'(z)| d\mu(z) &= \int_D G(z) d\mu(z) \\
&\leq C \int_D H^s(M^t G)(z) d\mu(z) \\
&\leq C \int_D M^t G(z) H^s\mu(z) da(z) \\
&= C \int_D \rho(z) M^t G(z) H^s\mu(z) \frac{da(z)}{1-|z|} \\
&\leq C ||\rho M^t G||_{T_2^1} ||H^s\mu||_{T_2^\infty},
\end{aligned}$$

which implies the desired inequality since $H^s\mu \leq H^t\mu$.

We consider next the case where $1/2 < p \leq 1$. Using the atomic decomposition for T^p_∞ (see [CMS]) it suffices to prove that there is a constant C such that for any arc I contained in T with $|I| \leq \frac{1}{4}$ and any function b supported in $T(I)$ and bounded by 1 the inequality

$$\int_T |L_\mu b|^p d\theta \leq C|I|$$

is verified. Write the integral on the left as a sum

$$\int_T = \int_{3I} + \int_{T-3I},$$

where CI is an interval with the same center as I and C times the radius. Then by the result proved above in the case $p = 2$

$$\begin{aligned} |\int_{3I}| &\leq C(\int_{3I} |L_\mu b|^2 d\theta)^{\frac{p}{2}} (|I|)^{1-\frac{p}{2}} \\ &\leq C(||b||^2_{T^2_2})^{\frac{2}{p}} (|I|)^{1-\frac{p}{2}} \\ &\leq C|I|^{\frac{2}{p}} |I|^{1-\frac{p}{2}} = C|I| \end{aligned}$$

which is the needed estimate.

For the second integral, let η be the center of I and estimate from (4) that for $\zeta \in T - 3I$

$$|L_\mu b(\zeta)| \leq |1 - \zeta\bar{\eta}|^{-2} \int_{T(I)} d\mu.$$

Observe that for all $z \in D$, $1 \leq CH^t 1(z)$ for a constant C depending only on t. Furthermore, if $t < 1$, then there is a constant C, again depending only on t such that the set $\{w \in D : |z - w| \leq t(1 - |z|) \quad \text{and} \quad z \in T(I)\}$ is contained in $T(CI)$. These two facts together with Fubini's theorem show that

$$\begin{aligned} \int_{T(I)} d\mu &\leq C \int_{T(I)} H^t 1\, d\mu \\ &\leq C \int_{T(CI)} H^t \mu(z) da(z) \\ &\leq C(\int_{T(CI)} (1 - |z|) da(z))^{\frac{1}{2}} (\int_{T(CI)} (H^t \mu(z))^2 \frac{da(z)}{1 - |z|})^{\frac{1}{2}} \\ &\leq C|I|^{\frac{3}{2}} |I|^{\frac{1}{2}} = |I|^2, \end{aligned}$$

where we have used Holder's inequality to obtain the third estimate and the fact that $H^t_\mu \in T^\infty_2$ to obtain the fourth estimate. Thus, for $\zeta \in T - 3I$,

$$|L_\mu b(\zeta)| \leq C|I|^2 |1 - \zeta\bar{\eta}|^{-2}.$$

Taking p^{th} powers and integrating over $T - 3I$ gives the needed estimate.

Suppose now that μ is a positive measure and L_μ defines a bounded operator from T^p_∞ into H^p. Consider first the case where $1 < p < \infty$. By duality we may start with (5) and interchange the order of integration as before to see that there is a constant $C = C(p)$ such that

$$|\int_D g(z) f'(z) d\mu(z)| \leq C||g||_{T^p_\infty} ||f||_{H^{p'}}$$

and it follows from this and Corollary B that for all $g \in T^p_\infty$, $F \in H^p$ and all $\psi \in BMOA$

$$\int_D |g(z)F(z)\psi'(z)|d\mu(z) \leq C||g||_{T^p_\infty}||F||_{H^{p'}}||\psi||_{BMOA}.$$

Therefore Theorem 1 yields the inequality

$$\int_D |H'(z)|d\mu(z) \leq C||H||_{H^1}$$

for all $H \in H^1$ and the result follows from Theorem 1 of [Lu].

Consider now the case where $\frac{1}{2} < p \leq 1$. Let I be an arc on T with $|I| \leq \frac{1}{4}$. If L_μ is a bounded operator from T^p_∞ into H^p then there is a constant $C = C(p)$ independent of I such that $||T_\mu b||^p_{H^p} \leq C|I|$ for any continuous function b supported in $T(3I)$ bounded by 1. From this it follows that there is a constant $C = C(p)$ independent of I such that

$$\frac{1}{|I|}\int_I |\int_{T(3I)} \frac{d\mu(z)}{(1-e^{i\theta}\bar{z})^2}|^p d\theta \leq C. \tag{6}$$

By bounding the real part of the inside integral in (6) from below it can be shown that there is a number t, $0 < t < 1$, and a constant $C = C(t)$ such that

$$\mu(D_t(z)) \leq C(1-|z|)^2 \tag{7}$$

for all $z \in D$. Let

$$F(\zeta) = L_\mu 1(\zeta) = \int_D \frac{d\mu(z)}{(1-\zeta\bar{z})^2}.$$

□

Claim. The function F belongs to BMOA.

To see this, define F_I and G_I by

$$F_I(\zeta) = \int_{T(3I)} \frac{d\mu(z)}{(1-\zeta\bar{z})^2}$$

and $G_I = F - F_I$. The following lemma depends essentially on the estimate (7).

Lemma 4. *There is a constant C such that for all arcs I contained in T*

$$|G_I(\zeta) - G_I(\zeta')| \leq C$$

for all ζ, ζ' in I.

Proof. Let η be the center of I. There is a constant C independent of I such that with ζ and ζ' as in the lemma and $z \in D - T(3I)$

$$|\frac{1}{(1-\zeta\bar{z})^2} - \frac{1}{(1-\zeta'\bar{z})^2}| \leq C\frac{|I|}{|1-\eta\bar{z}|^3}.$$

It follows that

$$\begin{aligned}|G_I(\zeta) - G_I(\zeta')| &= |\int_{D-T(3I)} \frac{1}{(1-\zeta\bar{z})^2} - \frac{1}{(1-\zeta'\bar{z})^2} d\mu(z)| \\ &\le C \int_{D-T(3I)} \frac{|I|}{|1-\eta\bar{z}|^3} d\mu(z) \\ &\le C \int_{D-T(3I)} \frac{|I|}{|1-\eta\bar{z}|^3} da(z) \\ &\le C,\end{aligned}$$

where we used (7) to replace $d\mu$ with da in the third line. This proves the lemma.

It follows from Lemma 4 that for any arc I in T there is a constant C, independent of I, a function b_I and a constant c_I both bounded by C such that

$$G_I \chi_I = (c_I + b_I)\chi_I,$$

where χ_I is the characteristic function of I.

We can now prove the claim. Since $F = G_I + F_I$ it follows that

$$\begin{aligned}F\chi_I &= (G_I + F_I)\chi_I \\ &= (c_I + b_I)\chi_I + F_I\chi_I,\end{aligned}$$

and therefore $(F - c_I)\chi_I = b_I\chi_I + F_I\chi_I$. It follows from this and (6) that

$$\int_I |F - c_I|^p d\theta \le C|I|$$

and this proves the claim.

The same argument shows that if b is any bounded function, then there is a constant C such that

$$||L_\mu b||_{BMOA} \le (C + \mu(D))||b||_\infty.$$

Therefore, if $F \in H^1$ it follows that there is a constant C such that

$$|\int_T F\overline{L_\mu b} d\theta| \le C||F||_{H^1}||b||_\infty$$

for all $F \in H^1$. Interchanging the order of integration in the integration on the left hand side yields the inequality

$$\int_D |F'(z)| d\mu(z) \le C||F||_{H^1}$$

for all $F \in H^1$. The desired result follows now by Theorem 1 of [Lu]. □

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