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ON SMOOTHNESS OF CARRYING SIMPLICES

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ABSTRACT. We consider dissipative strongly competitive systems $\dot{x}_i = x_i f_i(x)$ of ordinary differential equations. It is known that for a wide class of such systems there exists an invariant attracting hypersurface Σ, called the carrying simplex. In this note we give an amenable condition for Σ to be a C^1 submanifold-with-corners. We also provide conditions, based on a recent work of M. Benaïm (*On invariant hypersurfaces of strongly monotone maps*, J. Differential Equations **136** (1997), 302–319), guaranteeing that Σ is of class C^{k+1}.

1. INTRODUCTION

We consider systems of ordinary differential equations (ODE's) of class (at least) C^1

$$\dot{x}_i = x_i f_i(x) \tag{E}$$

on the nonnegative orthant $C := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n\}$, $n \geq 3$.

We write $F_i(x) = x_i f_i(x)$, $F = (F_1, \dots, F_n)$. The symbol $DF = [\partial F_i / \partial x_j]_{i,j=1}^n$ stands for the derivative matrix of the vector field F. The local flow generated by (E) on C will be denoted by $\phi = \{\phi_t\}$. A subset $B \subset C$ is *invariant* [resp. *forward invariant*] if $\phi_t x \in B$ for all $(t, x) \in \mathbb{R} \times B$ [resp. for all $(t, x) \in [0, \infty) \times B$] for which $\phi_t x$ is defined. For $x \in C$, $B \subset C$ the symbols $\omega(x)$, $\alpha(x)$, $\omega(B)$, $\alpha(B)$ have their usual meanings (see e.g. Hale [3]). A point $x \in C$ is a *rest point* if $\phi_t x = x$ for each $t \in \mathbb{R}$ (alternatively, if $F(x) = 0$). An invariant subset B of a compact invariant set S is called an *attractor* (resp. a *repeller*) *relative to* S if there is a relative neighborhood U of B in S such that $\omega(U) = B$ (resp. $\alpha(U) = B$). For an attractor B relative to S, by the repeller *complementary* to B we understand the set $\{x \in S : \omega(x) \cap B = \emptyset\}$. The attractor *complementary* to a repeller R is defined in an analogous way.

System (E) is *dissipative* if there is a compact set $B \subset C$ such that for each bounded $D \subset C$ its ω-limit set $\omega(D)$ is a nonempty subset of B. By standard results on global attractors (see [3]), for a dissipative system (E) there exists a compact invariant set $\Gamma \subset C$ (the *global attractor* for (E)) such that $\omega(D) \subset \Gamma$ for each bounded $D \subset C$. Evidently, $0 \in \Gamma$.

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For $I \subset \{1, \dots, n\}$ denote

$$C_I := \{x \in C : x_i = 0 \text{ for } i \in I\},$$
$$C_I^\circ := \{x \in C_I : x_j > 0 \text{ for } j \notin I\},$$
$$\partial C_I := C_I \setminus C_I^\circ.$$

From the form of (E) it follows readily that any C_I, as well as ∂C_I and C_I°, is invariant. We denote by $(\mathrm{E})_I$ the restriction of system (E) to C_I. Instead of $C_\emptyset^\circ$, $\partial C_\emptyset$, we write C°, ∂C. I' means $\{1, \dots, n\} \setminus I$.

If system (E) is dissipative, so are all of its subsystems $(\mathrm{E})_I$. For each $I \subset \{1, \dots, n\}$, the global attractor Γ_I for $(\mathrm{E})_I$ equals $\Gamma \cap C_I$.

System (E) is called *strongly competitive* if $(\partial f_i / \partial x_j)(x) < 0$ for each $1 \le i, j \le n$, $i \ne j$, $x \in C$. A strongly competitive system is called *totally competitive* if $(\partial f_i / \partial x_i)(x) < 0$ for $1 \le i \le n$, $x \in C$. Such systems describe a community of n interacting species where the growth of each species inhibits the growth of any other.

Throughout the rest of the paper the standing assumption will be:

(E) *is a C^1 dissipative strongly competitive system of ODE's satisfying the following:*

1. $\{0\}$ *is a repeller relative to* Γ.
2. *At each rest point $x \in C \setminus \{0\}$ one has $(\partial f_i / \partial x_i)(x) < 0$ for $1 \le i \le n$.*

The following important result was established by M. W. Hirsch ([4]).

Proposition 1.1. *The attractor $\Sigma \subset \Gamma$ complementary to the repeller $\{0\}$ is homeomorphic via radial projection to the standard $(n-1)$-simplex $\Delta := \{x \in C : x_1 + \dots + x_n = 1\}$. Moreover, the global attractor Γ equals the convex hull of $\Sigma \cup \{0\}$.*

Following M. L. Zeeman [15], the invariant compact set Σ is referred to as the *carrying simplex* for (E). In the ecological interpretation, the carrying simplex can be thought of as expressing the balance between the growth of small populations ($\{0\}$ is a repeller) and the competition of large populations (dissipativity).

M. W. Hirsch in [4] asked about sufficient conditions for the carrying simplex Σ to be of class C^1. The time reverse flow $\{\phi_{-t}\}_{t \ge 0}$ restricted to the invariant set C° is strongly monotone and its derivative flow is strongly positive (for these terms see H. L. Smith's monograph [12]). Therefore, when (E) possesses a repeller $R \subset \Sigma \cap C^\circ$ relative to Σ we can utilize a powerful recent result of I. Tereščák [13] on nonmonotone manifolds to conclude that the repulsion basin $B(R) := \{x \in \Sigma^\circ : \alpha(x) \subset R\}$ is a C^1 hypersurface. However, even in that case Tereščák's theorem does not apply to the whole of Σ, for the time reverse flow fails to be strongly monotone on the boundary ∂C. Moreover, if we assume that (E) is permanent (a natural assumption from the applied viewpoint) then there is an attractor A having the whole C° as its attraction basin, hence its repulsion basin (relative to Σ) equals A. In his paper [10] the present author gave a fairly weak condition implying the C^1 smoothness of Σ. It was done, however, at the expense of making use (for $n \ge 5$) of Pesin's theory of invariant measurable families of embedded manifolds, which compels one to assume that f has Hölder continuous derivatives.

In this note we show that a well-known, robust, and readily testable condition (see (A)) is enough to conclude that Σ is C^1. Because our proofs exploit Oseledets' theory of Lyapunov exponents, it suffices to assume f is C^1 to get C^1 smoothness of Σ. Next, conditions are given, based on recent results of M. Benaïm [1], for the carrying simplex to possess higher order smoothness.

I would like to thank Michel Benaïm for sending me a preprint of [1].

2. Statement of main results

For $I \subset \{1, \dots, n\}$ put

$$\Sigma_I := C_I \cap \Sigma, \quad \Sigma_I^\circ := C_I^\circ \cap \Sigma, \quad \partial\Sigma_I := \partial C_I \cap \Sigma.$$

We will call Σ_I a *k-dimensional face* of Σ, where $k = n - 1 - \operatorname{card} I$. Evidently all Σ_I, as well as Σ_I° and $\partial\Sigma_I$, are invariant. For $I \subset \{1, \dots, n\}$, the face Σ_I is the carrying simplex for subsystem $(\mathrm{E})_I$. The 0-dimensional face $\Sigma_{i'}$ consists of a single rest point $x^{(i)} = (0, \dots, 0, x_i^{(i)}, 0, \dots, 0)$ with $x_i^{(i)} > 0$ (called the *i-th axial rest point*).

Let $V = \{v = (v_1, \dots, v_n) : v_i \in \mathbb{R}\}$ stand for the vector space of all free n-dimensional vectors (in particular, we write the tangent bundle of the orthant C as $TC = C \times V$). Depending on the context, $\|\cdot\|$ may mean the Euclidean norm of a vector, or the operator norm of a matrix, associated with the Euclidean norm. For $I \subset \{1, \dots, n\}$, we denote

$$V_I := \{v \in V : v_i = 0 \text{ for } i \in I\}.$$

For any two points x, $y \in C_I$, we write $x \le_I y$ if $x_i \le y_i$ for all $i \in I'$, and $x <_I y$ if $x \le_I y$ and $x \ne y$. Moreover, $x \ll_I y$ if $x_i < y_i$ for all $i \in I'$. For $I = \emptyset$ we write simply $\le$, $<$, $\ll$. The reversed symbols are used in the obvious way. As each $(C_I, \le_I)$ is a lattice, we can define, for $I \subset \{1, \dots, n\}$ with $\operatorname{card} I \le n - 1$

$$x^{[I]} := \bigvee_{i \in I'} x^{(i)},$$

where it is easy to see that $x^{[J]} <_I x^{[I]}$ for $I \subsetneq J$.

The following result probably belongs to the folklore in the theory of competitive systems, but I have not been able to locate its proof.

Lemma 2.1. *For each $I \subset \{1, \dots, n\}$ with $1 \le \operatorname{card} I \le n - 2$ we have $y <_I x^{[I]}$ for all $y \in \Sigma_I$.*

Proof. Suppose to the contrary that there is $y \in \Sigma_I$ not in the $<_I$ relation to $x^{[I]}$. Assume first that $y = x^{[I]}$, that is, $x^{[I]} \in \Sigma_I$. For $i \in I'$, $j \in I'$, $i \ne j$, we have $x_j^{[I]} > x_j^{(i)} = 0$. As $f_i(x^{(i)}) = 0$, it follows by strong competitiveness that $f_i(x^{[I]}) < 0$ for $i \in I'$. Therefore we have $F_i(x^{[I]}) = x_i^{[I]} f_i(x^{[I]}) < 0$ for all $i \in I'$. Consequently, $\phi_t x^{[I]} \ll_I x^{[I]}$ for $t > 0$ sufficiently small. But Σ_I is invariant, so $\phi_t x^{[I]} \in \Sigma_I$ for all $t > 0$. We have thus obtained two points in Σ_I related by $\ll_I$, which contradicts Lemma 2.5 in Hirsch [4]. Assume that $y \in \Sigma_I$ is not in the $\le_I$ relation to $x^{[I]}$. Take an index k for which $y_k > x_k^{[I]}$. Let $J \subset \{1, \dots, n\}$ stand for the set of those indices j for which $y_j = 0$. Evidently $k \in J'$ and $I \subset J$. We have $y \in \Sigma_I \cap C_J^\circ = \Sigma \cap C_I \cap C_J^\circ = \Sigma \cap C_J^\circ = \Sigma_J^\circ$. As a consequence, $y_j > x_j^{(k)} = 0$ for $j \in J'$, $j \ne k$, and $y_k > x_k^{[I]} = x_k^{(k)}$ (since $k \notin I$). But this means that $y \gg_J x^{(k)}$. As both these points are in Σ_J, this again is in contradiction to Lemma 2.5 in [4]. $\square$

We say (E) satisfies hypothesis (A) if

For each $1 \le i \le n$ one has $f_i(x^{[i]}) \ge 0$.

In light of the strong competitiveness, (A) can be equivalently formulated as:

For each $I \subset \{1, \dots, n\}$ *with* $1 \le \operatorname{card} I \le n-1$ *one has* $f_i(x^{[I]}) \ge 0$ *for* $i \in I$.

Hypothesis (A) is well known in the literature on mathematical ecology. Consider the Lotka–Volterra competitive system

$$\dot{x}_i = x_i\Big(b_i - \sum_{j=1}^{n} a_{ij} x_j\Big), \tag{2.1}$$

with $b_i > 0$, $a_{ij} > 0$. For (2.1) the i-th axial rest point is given by $x_i^{(i)} = b_i/a_{ii}$. It is easy to see that (A) is now equivalent to

$$b_i \ge \sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ij} \frac{b_j}{a_{jj}} \quad \text{for each } 1 \le i \le n.$$

We are now in a position to state our main result.

Theorem A. *Assume that* (E) *satisfies* (A). *Then the carrying simplex* Σ *is a* C^1 *submanifold-with-corners neatly embedded in* C.

For submanifolds-with-corners their neat embeddings, see [10].

We now state some consequences of hypothesis (A). System $(\mathrm{E})_I$ is called *permanent* if there is $\epsilon > 0$ such that $\liminf_{t\to\infty} \rho(\phi_t x, \partial C_I) \ge \epsilon$ for each $x \in C_I^\circ$, where ρ stands for the Euclidean distance between a point and a set.

Proposition 2.2. *If* (A) *is satisfied, then each of the subsystems* $(\mathrm{E})_I$ *is permanent.*

Proof. In order not to encumber our presentation with too many subscripts, we prove the assertion for $I = \emptyset$, that is, for system (E) only. For each i, $1 \le i \le n$, we have as a result of strong competitiveness and Lemma 2.1 that $f_i(x) > 0$ for all $x \in \Sigma_{i'}$. Now take a neighborhood U_i of $\Sigma_{i'}$ in C of the form

$$U_i = \{(x_1, \dots, x_n) : 0 \le x_i < \epsilon_i, (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \tilde{U}_i\},$$

where $\epsilon_i > 0$ and a relative neighborhood $\tilde{U}_i$ of $\Sigma_{i'}$ in $C_{i'}$ are so small that $f_i(x) > 0$ for all $x \in U_i$. As Γ is the global attractor for (E) and Σ is the attractor relative to Γ complementary to $\{0\}$, there exists a forward invariant neighborhood U of Σ in C with the property that $\phi_t x \in U$ for $x \in C \setminus \{0\}$ and sufficiently large t. Also, U can be taken so small that all the sets $\{x \in U : x_i < \epsilon_i\}$ are contained in U_i. Now observe that for t so large that $\phi_t x$ belongs to U one has

$$\frac{d(\phi_t x)_i}{dt} = F_i(\phi_t x) = x_i f_i(\phi_t x) > 0$$

as long as $(\phi_t x)_i < \epsilon_i$. From this it readily follows that $\liminf_{t\to\infty} \rho(\phi_t x, \Sigma_{i'}) \ge \epsilon_i$ for any $x \in C^\circ$. $\square$

In view of results on attractors contained in Hale [3] we have the following.

Lemma 2.3. *Under the assumptions of Proposition 2.2, for each* $I \subset \{1, \dots, n\}$ *the invariant compact set* $\partial\Sigma_I$ *is a repeller relative to* Σ_I.

For $I \subset \{1, \dots, n\}$ denote by A_I the attractor (relative to Σ_I) complementary to $\partial\Sigma_I$. As A_I can be viewed as the global attractor for the semiflow $\{\phi_t\}_{t\ge 0}$ restricted to the connected metric space Σ_I°, a result of Gobbino and Sardella (Thm. 3.1 in [2]) yields that A_I is connected.

The ecological interpretation of the property described in Proposition 2.2 is as follows. In each subcommunity none of the species goes extinct, and invasion of a proper subcommunity by others causes the populations of the previously present species to shrink due to the larger amount of competition.

Before formulating sufficient conditions for Σ to be of class C^{k+1} we need to introduce some notation (we follow Benaïm's paper [1]). For $x \in A_I$, $I \subset \{1, \dots, n\}$ with $\operatorname{card} I \leq n-2$, we denote by $\lambda(x)$ the largest eigenvalue of the symmetrization of the matrix $(-DF^I(x))$, where $DF^I := [\partial F_i/\partial x_j]_{(i,j)\in I' \times I'}$. Further, $d(x)$ stands for the square root of

$$\min_{\substack{i \neq j \\ i,j \notin I}} \frac{\partial F_i}{\partial x_j}(x) \frac{\partial F_j}{\partial x_i}(x).$$

Put $\lambda_I := \sup\{\lambda(x) : x \in A_I\}$ and $d_I := \inf\{d(x) : x \in A_I\}$.

We say that (E) satisfying (A) fulfills (C) if for each I with $0 \leq \operatorname{card} I \leq n-2$ any one of the conditions (C1) or (C2) holds:

(C1) $k \sup\{\|DF^I(x)\| : x \in A_I\} < 2(k+1)d_I$.

(C2) $k\lambda_I < 2(k+1)d_I$.

Theorem B. *Assume that a C^{k+1} system* (E) *satisfies* (A) *and* (C). *Then the carrying simplex Σ is a C^{k+1} submanifold-with-corners.*

3. Proof of Theorem A

Let $\mathbb{S}$ be the $(n-1)$-dimensional sphere $\{v \in V : \|v\| = 1\}$. For a vector subspace W of V and $0 \leq k \leq \dim W$, the symbol $\mathbb{G}_k W$ denotes the compact metrizable space of all k-dimensional vector subspaces of W, endowed with the standard topology: for any two Z_1, $Z_2 \in \mathbb{G}_k W$, their distance is defined as the Hausdorff distance between $Z_1 \cap \mathbb{S}$ and $Z_2 \cap \mathbb{S}$.

The linearization of (E) generates on TC a linear skew-product (local) flow $(\phi_t x, D\phi_t(x)v)$, where $D\phi_{t_0}(x)v_0$ is the value at time t_0 of the solution of the variational equation $\dot{\xi} = DF(\phi_t x)\xi$ with initial condition $\xi(0) = v_0$.

For a linear subset $\mathcal{C}$ of the product bundle $B \times W$, where $B \subset \Sigma$ and W is a vector subspace of V, we will denote by $\mathcal{C}_x$ the set of all those $v \in W$ such that $(x, v) \in \mathcal{C}$ (in other words, $\{x\} \times \mathcal{C}_x$ is the fiber of $\mathcal{C}$ over x). A linear subset $\mathcal{C}$ of $B \times W$ is called *invariant* if for each $(x, v) \in \mathcal{C}$ and each $t \in \mathbb{R}$ one has $(\phi_t x, D\phi_t(x)v) \in \mathcal{C}$.

Denote the set of all ergodic measures supported on a compact invariant $B \subset \Sigma$ by $\mathbf{M}_{\mathrm{erg}}(B)$. The multiplicative ergodic theorem of Oseledets (see e.g. Mañé [8]) assures us that if $B \times W$ is an invariant bundle, then for each $m \in \mathbf{M}_{\mathrm{erg}}(B)$ there exist an invariant m-measurable set $B_{\mathrm{reg}} \subset B$ (the set of *regular points*), a collection $\mathcal{C}_1, \dots, \mathcal{C}_l$ of invariant linear subsets given by m-measurable maps $B_{\mathrm{reg}} \ni x \mapsto (\mathcal{C}_k)_x \in \mathbb{G}_{d_k} W$ (the *Oseledets decomposition*) and a collection $\Lambda_1 < \cdots < \Lambda_l$ of reals (*Lyapunov exponents*) such that

1. $W = \bigoplus_{k=1}^{l} (\mathcal{C}_k)_x$ for $x \in B_{\mathrm{reg}}$,
2. $$\lim_{t \to \pm\infty} \frac{\log\|D\phi_t(x)v\|}{t} = \Lambda_k$$
   for $1 \leq k \leq l$, $x \in B_{\mathrm{reg}}$ and $v \in (\mathcal{C}_k)_x$.

Lemma 3.1. *For each* $m \in \mathbf{M}_{\mathrm{erg}}(\Sigma)$ *there is* $I = I(m) \subsetneq \{1, \dots, n\}$ *such that the support* $\operatorname{supp} m$ *of* m *is contained in* A_I.

Proof. By ergodicity of m and invariance of all Σ_I°, there is precisely one $I \subset \{1, \dots, n\}$ such that $m(\Sigma_I^\circ) = 1$ and $m(\partial\Sigma_I) = 0$. Further, as points from $\Sigma_I^\circ \setminus A_I$ are wandering (relative to Σ), one has $m(\Sigma_I^\circ \setminus A_I) = 0$. □

Fix $m \in \mathbf{M}_{\mathrm{erg}}(\Sigma_I)$ with $m(\Sigma_I^\circ) = 1$, and put $\mathcal{B} := \Sigma_I \times V_I$, $\mathcal{B}^{(i)} := \Sigma_I \times V_{I \setminus i}$, $i \in I$. Evidently, $\mathcal{B}$ is a subbundle of $\mathcal{B}^{(i)}$ of codimension one. From the structure of system (E) it follows that the bundles $\mathcal{B}$, $\mathcal{B}^{(i)}$ are invariant. Denote by $\Lambda_1 < \Lambda_2 \dots < \Lambda_l$ the Lyapunov exponents on $\mathcal{B}$ for the ergodic measure m (we will call them the *internal Lyapunov exponents* for m). Among the Lyapunov exponents on $\mathcal{B}^{(i)}$ there is one (denoted by $\lambda^{(i)}(m)$) corresponding to the measurable linear set $\mathcal{C}_k^{(i)} \subset \mathcal{B}^{(i)}$ such that $(\mathcal{C}_k^{(i)})_x \subsetneq V_I$ for m-a.e. $x \in \Sigma_I^\circ$. We will refer to $\lambda^{(i)}(m)$ as the i-th *external Lyapunov exponent* for m (this terminology is modeled on Hofbauer's [6]).

The following result was essentially proved in the author's paper [10] (except for terminology).

Theorem 3.2. *Assume that for each* $m \in \mathbf{M}_{\mathrm{erg}}(\partial\Sigma)$ *all its external Lyapunov exponents are nonnegative. Then the following hold:*

1. *The carrying simplex* Σ *is a* C^1 *submanifold-with-corners neatly embedded in* C.
2. *There are* $\mu > 0$ *and an invariant one-dimensional subbundle* $\mathcal{S}$ *of* $\Sigma \times V$ *such that for each* $m \in \mathbf{M}_{\mathrm{erg}}(\Sigma)$ *one has*
   (a) $\Sigma \times V = T\Sigma \oplus \mathcal{S}$, *where* $T\Sigma$ *denotes the tangent bundle of* Σ, *and* $(\mathcal{C}_1)_x \subset \mathcal{S}$ *for* m*-a.e.* $x \in \Sigma$.
   (b) Λ_1 *is internal.*
   (c) $\Lambda_1 \le -\mu$.

In the present section we make use of part 1 of Theorem 3.2 only.

In view of the above result, we need to prove only the following.

Proposition 3.3. *Under the assumptions of Theorem A, for each* $m \in \mathbf{M}_{\mathrm{erg}}(\partial\Sigma)$ *all its external Lyapunov exponents are nonnegative.*

Proof. Fix a measure $m \in \mathbf{M}_{\mathrm{erg}}(\Sigma_I)$ with $m(\Sigma_I^\circ) = 1$, and an index $i \in I$. By Lemma 3.1, $\operatorname{supp} m \subset A_I$. Take a regular point $x \in \operatorname{supp} m$ and a vector $v \in (\mathcal{B}^{(i)})_x \setminus V_I$ such that its i-th coordinate v_i is positive. As $(\partial F_i/\partial x_j)(\phi_t x) = 0$ for $j \ne i$, and $(\partial F_i/\partial x_i)(\phi_t x) = f_i(\phi_t x)$, it follows that the i-th coordinate $(D\phi_t(x)v)_i$ is the solution of the (nonautonomous) scalar linear ODE $\dot\eta = f_i(\phi_t x)\eta$ with initial condition $\eta(0) = v_i$. By strong competitiveness and Lemma 2.1, f_i is positive on the compact invariant set $A_I \subset \Sigma_I$, hence there is $M > 0$ such that $f_i(\phi_t x) \ge M$ for all $(t, x) \in \mathbb{R} \times A_I$. The standard theory of differential inequalities yields

$$\liminf_{t\to\infty} \frac{\log (D\phi_t(x)v)_i}{t} \ge M.$$

$\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$, therefore for all $t \in \mathbb{R}$ we have $\|D\phi_t(x)v\| \ge (D\phi_t(x)v)_i$. By regularity of x we derive

$$\lim_{t\to\infty} \frac{\log \|D\phi_t(x)v\|}{t} = \lambda^{(i)}(m) \ge M > 0.$$

□

4. Proof of Theorem B

We begin by stating a result which is an adaptation of a theorem of M. Benaïm.

Theorem 4.1. *Assume that a C^{k+1}, $k = 1, \dots,$ system* (E) *satisfies the following:*

1. *For each $m \in \mathbf{M}_{\mathrm{erg}}(\partial\Sigma)$ all external Lyapunov exponents are nonnegative.*
2. *There is $\eta > 0$ such that for each $m \in \mathbf{M}_{\mathrm{erg}}(\Sigma)$ the inequality*

$$\Lambda_1(m) - (k+1)\Lambda_2(m) < -\eta \tag{4.1}$$

holds, where $\Lambda_1(m)$ and $\Lambda_2(m)$ denote respectively the smallest and the second smallest Lyapunov exponents (on $\Sigma \times V$) for m.

Then the carrying simplex Σ is a C^{k+1} submanifold-with-corners.

Indication of proof. Theorem 3.2.2 asserts that the tangent bundle TC restricted to Σ invariantly decomposes as the Whitney sum $T\Sigma \oplus \mathcal{S}$, and for each $m \in \mathbf{M}_{\mathrm{erg}}(\Sigma)$ the smallest Lyapunov exponent $\Lambda_1(m) \le -\mu < 0$ is the exponential growth rate of a vector from $\mathcal{S}$, while any of the remaining Lyapunov exponents is the exponential growth rate of a vector tangent to Σ. This, together with (4.1), gives, with the help of Prop. 3.3 in [1] (based on a result of S. Schreiber [11]), that there are $c \ge 1$, $\alpha > 0$ and $\beta > 0$ such that

$$\|D\phi_t(x)v\| \le ce^{-\alpha t}\|v\| \quad \text{for } t \ge 0,\ (x,v) \in \mathcal{S},$$

and

$$\frac{\|D\phi_t(x)v\|}{\|D\phi_t(x)w\|^{k+1}} \le ce^{-\beta t} \quad \text{for } t \ge 0,\ x \in \Sigma,\ v \in (\mathcal{S})_x,\ w \in T_x\Sigma \setminus \{0\}.$$

The rest of the proof consists in applying the C^{k+1} section theorem of Hirsch, Pugh and Shub [5], as in the proof of Thm. 3.4 in [1]. □

As a consequence of the above theorem and Proposition 3.3, we will have Theorem B once we prove the following.

Proposition 4.2. *Assume that a C^{k+1} system* (E) *satisfies* (A) *and* (C)*. Then there exists $\eta > 0$ such that for each $m \in \mathbf{M}_{\mathrm{erg}}(\Sigma)$ the inequality* (4.1) *holds.*

Proof. Take $m \in \mathbf{M}_{\mathrm{erg}}(\Sigma)$, and let $I \subset \{1, \dots, n\}$ be such that $m(\Sigma_I^\circ) = 1$. From Lemma 3.1 we have $\operatorname{supp} m \subset A_I$. By results contained in Sections 3 and 4 of [1], it follows that under assumption (C) there is $\eta_I > 0$ such that

$$\Lambda_1^*(m) - (k+1)\Lambda_2^*(m) < -\eta_I$$

for all m supported on A_I, where $\Lambda_1^*(m)$ [resp. $\Lambda_2^*(m)$] stands for the smallest [resp. second smallest] internal Lyapunov exponent for m. Theorem 3.2.2 gives $\Lambda_1^*(m) = \Lambda_1(m)$. Denote by $\lambda_{\min}$ the smallest external Lyapunov exponent for m. If $\lambda_{\min} \ge \Lambda_2^*(m)$, then $\Lambda_2^*(m) = \Lambda_2(m)$ and the inequality (4.1) is satisfied with η_I. Assume that $\lambda_{\min} < \Lambda_2^*(m)$. Applying Theorem 3.2.2 and Proposition 3.3, we obtain $\Lambda_1(m) \le -\mu < 0 \le \lambda_{\min} = \Lambda_2(m)$. Consequently, $\Lambda_1(m) \le -\mu < 0 \le (k+1)\Lambda_2(m)$. It suffices to put

$$\eta := \min\{\mu, \eta_I : I \subset \{1, \dots, n\}, \operatorname{card} I \le n-1\}.$$

□

5. Discussion

Remark 5.1. In formulating our results, we preferred that the assumptions be easily tractable rather than the weakest possible or that they cover a wide range of applications. In fact, they can be substantially weakened, as the following example shows.

A celebrated Lotka–Volterra system due to May and Leonard [9] has the form

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1 - \alpha x_2 - \beta x_3), \\ \dot{x}_2 &= x_2(1 - \beta x_1 - x_2 - \alpha x_3), \\ \dot{x}_3 &= x_3(1 - \alpha x_1 - \beta x_2 - x_3), \end{aligned} \tag{5.1}$$

with $0 < \beta < 1 < \alpha$ and $\alpha + \beta > 2$. It is easily verified that (5.1) is dissipative, totally competitive and has five rest points on C: 0 (repelling), y with $y_i = 1/(1 + \alpha + \beta)$ and three axial ones $x^{(i)}$ with $x^{(i)}_i = 1$. Furthermore, $\partial\Sigma$ is an attractor relative to Σ with $\{y\}$ as its complementary repeller (see pp. 67–68 in the book [7] by Hofbauer and Sigmund). As a consequence, $\mathbf{M}_{\mathrm{erg}}(\Sigma) = \{\delta_y, \delta_{x^{(1)}}, \delta_{x^{(2)}}, \delta_{x^{(3)}}\}$. The Lyapunov exponents for the Dirac delta on a rest point x are simply the real parts of the eigenvalues of $DF(x)$. At y the smallest Lyapunov exponent is negative and the remaining ones are positive (see [7]). A simple calculation shows that at an axial rest point the unique internal Lyapunov exponent equals -1 and the external exponents are $1 - \beta > 0$ and $1 - \alpha < 0$.

Assume now that $\alpha < 2$. Then for each ergodic measure on Σ the smallest Lyapunov exponent -1 is internal. Applying ideas of the author's earlier paper [10] we prove that the carrying simplex Σ for (5.1) is a C^1 submanifold-with-corners. Observe that if $1 < \alpha < 1 + 1/l$ for some $l = 2, \dots$, then $-1 - l(1 - \alpha)$ is negative and one can deduce along the lines of the proof of Theorem 4.1 to conclude that Σ is of class C^l.

Remark 5.2. The systems (E) satisfying (A) [resp. (A) and (C)] are robust in the sense that if we perturb f in a neighborhood of Σ in the C^1 [resp. in the C^{k+1}] topology, then the perturbed system possesses a carrying simplex of class C^1 [resp. C^{k+1}] (and each of its subsystems $(\mathrm{E})_I$ is permanent). This can be proved by reasoning similar to that in the proof of Cor. 4.3 in [1].

Remark 5.3. In principle, results contained in Section 3 should carry over to the case where we allow f to depend periodically on t, although finding an analog of (A) might be tricky (for time-periodic Lotka–Volterra strongly competitive systems, compare e.g. [14]).

References

[1] M. Benaïm, *On invariant hypersurfaces of strongly monotone maps*, J. Differential Equations **137** (1997), 302–319. MR **98d:**58114

[2] M. Gobbino and M. Sardella, *On the connectedness of attractors for dynamical systems*, J. Differential Equations **133** (1997), 1–14. CMP 97:06

[3] J. K. Hale, *Asymptotic behavior of dissipative systems*, Math. Surveys Monogr. **25**, American Mathematical Society, Providence, R.I., 1988. MR **89g:**58059

[4] M. W. Hirsch, *Systems of differential equations which are competitive or cooperative. III. Competing species*, Nonlinearity **1** (1988), 51–71. MR **90d:**58070

[5] M. W. Hirsch, C. C. Pugh and M. Shub *Invariant manifolds*, Lecture Notes in Math. **583**, Springer, Berlin–New York, 1977. MR **58:**18595

[6] J. Hofbauer, *An index theorem for dissipative semiflows*, Rocky Mountain J. Math. **20** (1990), 1017–1031. MR **92b:**58203

[7] J. Hofbauer and K. Sigmund, *The theory of evolution and dynamical systems*, London Mathematical Society Student Texts **7**, Cambridge University Press, Cambridge, 1988. MR **91h:**92019

[8] R. Mañé, *Ergodic theory and differentiable dynamics* (translated from the Portuguese by S. Levy), Ergeb. Math. Grenzgeb. (3) **8**, Springer, Berlin–New York, 1987. MR **88c:**58040

[9] R. M. May and W. J. Leonard, *Nonlinear aspects of competition between three species*, SIAM J. Appl. Math. **29** (1975), 243–253. MR **52:**12853

[10] J. Mierczyński, *The C^1 property of carrying simplices for a class of competitive systems of ODEs*, J. Differential Equations **111** (1994), 385–409. MR **95g:**34066

[11] S. J. Schreiber, *Expansion rates and Lyapunov exponents*, Discrete Contin. Dynam. Systems **3** (1997), 433–438. MR **98c:**58096

[12] H. L. Smith, *Monotone dynamical systems. An introduction to the theory of competitive and cooperative systems*, Math. Surveys Monogr. **41**, Amer. Math. Soc., Providence, R.I., 1995. MR **96c:**34002

[13] I. Tereščák, *Dynamics of C^1 smooth strongly monotone discrete-time dynamical systems*, preprint.

[14] A. Tineo, *An iterative scheme for the N-competing species problem*, J. Differential Equations **116** (1995), 1–15. MR **95m:**92023

[15] M. L. Zeeman, *Hopf bifurcations in competitive three-dimensional Lotka–Volterra systems*, Dynam. Stability Systems **8** (1993), 189–217. MR **94j:**34044

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