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ON SUSPENSIONS OF NONCONTRACTIBLE COMPACTA OF TRIVIAL SHAPE

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Dedicated to the memory of the Teacher Hilol Karimov

Abstract. We prove that: (i) There exists a 2-dimensional noncontractible cohomologically locally connected cell-like compactum whose reduced suspension is a contractible ANR; (ii) If the suspension ΣX of a compactum X is contractible, then X is weakly contractible.

1. Introduction

In 1904 Poincaré constructed the first example of a polyhedral homological 3-sphere with nontrivial fundamental group [12]. The complement of an open star of a vertex in this space is a noncontractible acyclic finite polyhedron P. By the Mayer-Vietoris exact sequences and the Van Kampen Theorem it follows that the suspension ΣP of this polyhedron is an acyclic space with the trivial fundamental group. It follows by the Hurewicz Theorem that therefore the suspension ΣP has all homotopy groups trivial and is hence a contractible space. Complex P is a noncontractible acyclic compactum. Every cell-like space is acyclic in Čech cohomology and every contractible compactum is clearly cell-like. So there is a natural question: Does there exist a noncontractible cell-like compactum whose suspension is contractible? [10, Problem 677]. The purpose of this paper is to prove the following related results:

Theorem (1.1). *There exists a* 2*-dimensional noncontractible cohomologically locally connected cell-like compactum whose reduced suspension is a contractible ANR.*

The example constructed in Theorem (1.1) is not weakly contractible and hence, according to Theorem (1.2) its unreduced suspension is not contractible.

Theorem (1.2). *If the suspension ΣX of a compactum X is contractible, then X is weakly contractible.*

2. Preliminaries

To fix terminology we give some definitions. The *suspension* of a space Z, denoted by ΣZ, is defined as the quotient space of the product $Z \times I$ of Z and the

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segment $I = [0,1]$ in which $Z \times 0$ is identified to point v_0 and $Z \times 1$ is identified to another point v_1. Points v_0 and v_1 are called the *vertices* of ΣZ. The *reduced suspension* of space Z relative to the point $z_0 \in Z$, denoted by $\widetilde{\Sigma} Z(\operatorname{rel} z_0)$, is defined as the quotient space of $Z \times I$ in which $(Z \times 0) \cup (z_0 \times I) \cup (Z \times I)$ is identified to a single point. For any point $(z,\tau) \in Z \times I$, we use $[z,\tau]$ to denote the corresponding point of ΣZ. For any map $f : Z' \to Z$, the induced map $\Sigma f : \Sigma Z' \to \Sigma Z$ is defined by $(\Sigma f)([z,\tau]) = [f(z),\tau]$. The space ΣZ is not always homotopically equivalent to $\widetilde{\Sigma} Z(\operatorname{rel} z_0)$. For example, let Z be the one-point compactification N^* of a countable discrete space. Then ΣN^* and $\widetilde{\Sigma} N(\operatorname{rel} *)$ are not homotopically equivalent (they have countable and uncountable fundamental groups, respectively).

Space Z is said to be *weakly contractible* (wc), if for every point $z_0 \in Z$, there exists a neighborhood V of the point z_0 such that the inclusion $V \subset Z$ is homotopically trivial. So every contractible or locally contractible space and every suspension is weakly contractible. To every mapping $g : Y \to \Sigma Z$ there is associated a canonical set-valued mapping $\phi_g : Y \to Z$ and a function $\psi_g : Y \to I$ defined by the equality $g(y) = [\phi_g(y), \psi_g(y)]$, $y \in Y$. The mapping $g : \Sigma Z' \to \Sigma Z$ is said to be *flat* if for every two points $[z'_1, \tau'_1]$ and $[z'_2, \tau'_2]$ with $\tau'_1 = \tau'_2$, the following equality holds: $\psi_g([z'_1, \tau'_1]) = \psi_g([z'_1, \tau'_2])$. Homotopy $H : \Sigma Z' \times I \to \Sigma Z$ is called a *flat homotopy* if for every fixed $t_0 \in I$, $H(\ ,t_0) : \Sigma Z' \to \Sigma Z$ is a flat mapping.

Let $\mathcal{K} = \{K_{i-1} \overset{f_{i-1}}{\longleftarrow} K_i\}_{i \in \mathbb{N}}$ be a sequence of compact polyhedron K_i, where K_0 is a point. Let $Y = \varprojlim \mathcal{K}$, and let $\{C(f_0, f_1, \dots, f_n)\}_{n \in \mathbb{N}}$ be the associated (with $\mathcal{K}$) sequence of finite CW-complexes. Now let $C(f_0, f_1, f_2, \dots)$ be the infinite mapping cylinder (see, e.g. [13]) and let $Y_{\mathcal{K}}$ be the natural compactification of $C(f_0, f_1, f_2, \dots)$ by Y, attached as a Z-set. The following is well known (see, e.g. [3]):

Proposition (2.1). *Space $Y_{\mathcal{K}}$ is an absolute retract.*

Let $X_{\mathcal{K}}$ be the one-point compactification by some point k of $C(f_0, f_1, f_3, \dots)$. Obviously, $X_{\mathcal{K}}$ is homeomorphic to the quotient space $Y_{\mathcal{K}}/Y$. In the case when Y is acyclic and $Y_{\mathcal{K}}$ is finite-dimensional it follows by the Whitehead theorem in shape theory [9] that $\operatorname{Sh}(X_{\mathcal{K}}) = \operatorname{Sh}(Y_{\mathcal{K}})$. Therefore by Proposition (2.1) we have:

Proposition (2.2). *Compactum $X_{\mathcal{K}}$ is a cell-like space.*

Remark (2.3). If $\mathcal{K}'$ is another inverse sequence for which $\operatorname{Sh} Y' = \operatorname{Sh} Y$, then $X_{\mathcal{K}'}$ is homotopically equivalent to $X_{\mathcal{K}}$ (see, e.g. [13, p. 375]).

3. Proof of Theorem (1.1)

Consider the Case-Chamberlin inverse sequence $\mathcal{P}$ [2]:

$$P_0 \overset{f_0}{\longleftarrow} P_1 \overset{f_1}{\longleftarrow} P_2 \overset{f_2}{\longleftarrow} \cdots$$

where P_0 is a point, P_i is a bouquet of two circles $S^1 \vee S^1$, for every $i > 0$, $f_i = f : S^1 \vee S^1 \to S^1 \vee S^1$ is a continuous mapping which maps the natural generators a and b of the fundamental group $\pi_1(S^1 \vee S^1)$ to the commutators $[a,b]$ and $[a^2,b^2]$ of $\pi_1(S^1 \vee S^1)$, respectively. Since the inverse sequence $\mathcal{P}$ is acyclic and $\dim C(f_0, f_1, f_2, \dots) = 2$, we have by Proposition (2.2) the following:

Proposition (3.1). *$X_{\mathcal{P}}$ is a 2-dimensional cell-like compactum.*

Next, we prove the following lemma.

Lemma (3.2). *Let X be a space with finitely generated Čech cohomology and suppose that for every point $x \in X$, there exists a basis of neighborhoods $\mathcal{U}_x = \{U_\alpha\}_{\alpha \in A(x)}$ in X such that for every $\alpha \in A(x)$, the boundary $\operatorname{Fr} U_\alpha$ also has finitely generated Čech cohomology. Then X is a cohomologically locally connected space.*

Proof. Consider the Mayer-Vietoris exact sequence:

$$\cdots \to \check{H}^n(X) \to \check{H}^n(X \setminus U_\alpha) \oplus \check{H}^n(\overline{U}_\alpha) \to \check{H}^n(\operatorname{Fr} U_\alpha) \to \cdots .$$

It follows that $\check{H}^n(\overline{U}_\alpha)$ is finitely generated. Now, by the continuity property of the Čech cohomology there exists a neighborhood $O_\alpha \subset U_\alpha$ such that the restriction $\check{H}^n(\overline{U}_\alpha) \to \check{H}^n(O_\alpha)$ is trivial (for $n = 0$ consider the reduced cohomology). Therefore X is indeed a clc space. □

Condition that the Čech cohomology of X is finitely generated is essential as the examples $X = \widetilde{\Sigma} N^*(\operatorname{rel} *)$ of $X = N^*$ show. From Lemma (3.2) and by construction of $X_{\mathcal{P}}$ we can deduce the following:

Proposition (3.3). *Compactum $X_{\mathcal{P}}$ is a clc space.*

In order to prove that X is noncontractible we first need to prove the following two lemmas:

Lemma (3.4). *For every $n > 0$, the image of $a_n \in \pi_1(P_n)$ under the homomorphism $\pi_1(P_n) \xrightarrow{i_n} \pi_1(X_{\mathcal{P}} \setminus \{P_0\})$ is nonzero.*

Proof. Let Y_n be the quotient space of $X_{\mathcal{P}} \setminus P_0$ by $\overline{X_{\mathcal{P}} \setminus C(f_0, f_1, \ldots, f_n)}$ and let j_n be the composition of the natural homomorphisms $\pi_1(P_n) \to \pi_1(X_{\mathcal{P}} \setminus P_0) \to \pi_1(Y_n)$. Clearly, Y_n is homotopically equivalent to a finite CW complex whose fundamental group has the following presentation:

$$\begin{gathered}\langle a_1, b_1, a_2, b_2, \ldots, a_n, b_n : a_2[a_1, b_1], b_2[a_1^2, b_1^2], \ldots \\ a_n[a_{n-1}, b_{n-1}], b_n[a_{n-1}^2, b_{n-1}^2], [a_n, b_n], [a_n^2, b_n^2]\rangle.\end{gathered}$$

By the Tietze transformations this presentation reduces to one with a single relation, $\langle a, b : \alpha_n(a, b)\rangle$. The words $\alpha_n(a, b)$ are defined inductively. Namely, $\alpha_1(a, b) = [a, b]$ and $\alpha_{k+1}(a, b) = \alpha_k([a, b], [a^2, b^2])$, $k \in \mathbb{N}$. In this presentation $j_n(a_n)$ equals $\alpha_{n-1}(a, b)$ and its weight is $n - 1$ (cf. [7, 8]), hence $i_n(a_n) \neq 0$. □

Lemma (3.5). *Let X be a compact space and suppose there is a point $x_0 \in X$ such that $X \setminus \{x_0\}$ is locally contractible and there is a compact subset $P \subset X \setminus \{x_0\}$ such that no neighborhood $V \subset X \setminus P$ of x_0 is contractible in $X \setminus P$. Then X is not a wc space.*

Proof. Suppose to the contrary, that X is a wc space. Let V be a neighborhood of x_0 such that $i : V \hookrightarrow X$ is a contractible embedding. Any contraction which fixes x_0 will shrink sufficiently small neighborhood $W \subset V$ of the point x_0 in the complement of P. On the other hand, any contraction which moves x_0 will take a very small neighborhood of x_0 into a small neighborhood in $X \setminus (P \cup \{x_0\})$ where it will shrink it in the complement of P. Contradiction. □

Proposition (3.6). *The compactum $X_{\mathcal{P}}$ is not weakly contractible.*

Proof. Following Lemma (3.5), let X be $X_{\mathcal{P}}$, $x_0 = p$ be the compactification point of the Case-Chamberlin infinite mapping cylinder, and $P = P_0$ the first term of the Case-Chamberlin inverse sequence. Let V be any neighborhood of the point p in $X_{\mathcal{P}} \setminus \{P_0\}$. Then by Lemma (3.4) there exists a loop in V which is nontrivial in $X_{\mathcal{P}} \setminus \{P_0\}$. Applying Lemma (3.5), we conclude that $X_{\mathcal{P}}$ is not weakly contractible. □

Proposition (3.7). *The reduced suspension $\widetilde{\Sigma} X_{\mathcal{P}}(\operatorname{rel} p)$ of $X_{\mathcal{P}}$ is an AR.*

Proof. Consider the suspension $\Sigma\mathcal{P}$ of the Case-Chamberlin inverse sequence:

$$\Sigma P_0 \leftarrow \Sigma P_1 \leftarrow \Sigma P_1 \leftarrow \cdots$$

and its inverse limit ΣY. Obviously, $\widetilde{\Sigma} X_{\mathcal{P}}(\operatorname{rel} p) = Y_{\Sigma\mathcal{P}}/\Sigma Y$. By Proposition (2.1), $Y_{\Sigma\mathcal{P}}$ is an AR. By the Whitehead theorem [9], $\operatorname{Sh}(\Sigma Y) = \operatorname{Sh}(*)$. So $\widetilde{\Sigma} X_{\mathcal{P}}$ is a finite-dimensional cell-like image of the AR and therefore is also an AR [4, 6]. □

Theorem (1.1) is now a direct consequence of Propositions (3.1), (3.3), (3.6), and (3.7). □

4. Proof of Theorem (1.2)

Lemma (4.1). *Let Z' and Z be compact spaces, $\Sigma f : \Sigma Z' \to \Sigma Z$ be a homotopically trivial flat mapping and $H : \Sigma Z' \times I \to \Sigma Z$ a homotopy between Σf and a constant mapping. Suppose that for no τ', $\tau \in I$, the set $\{H([z', \tau'], t) | z' \in Z'\}$ contains both vertices v_0 and v_1 of ΣZ. Then there exists a flat homotopy $H' : \Sigma Z \times I \to \Sigma Z$ which connects Σf with the constant mapping.*

Proof. Let ϕ_H and ψ_H be the mappings corresponding to $H : \Sigma Z' \times I \to \Sigma Z$ as defined in section 2. Fix the numbers τ' and t. Let $a(\tau', t)$ and $b(\tau', t)$ be the minimum and the maximum of the function $\psi_H([\ ,\tau'], t) : Z' \to I$. Define the mapping $H' : \Sigma Z' \times I \to \Sigma Z$ by the following formula:

$$H'([z', \tau'], t) = \left[\phi_H([z', \tau'], t), \frac{a(\tau', t)}{1 + a(\tau', t) - b(\tau', t)}\right].$$

It is not difficult to check that H' is well-defined and that it has the required properties. □

Let $H : \Sigma Z' \times I \to \Sigma Z$ be a flat homotopy. Then there corresponds to H a mapping $h : I^2 \to I$ for which $h(\tau', t) = \psi_H([z', \tau'], t)$, $z' \in Z'$. Let $p_i : I^2 \to I$, $i = 1, 2$, be the projections, defined by $p_1(\tau, t) = \tau$ and $p_2(\tau, t) = t$, respectively.

Proof of Theorem (1.2). Suppose that Z is a non-wc compactum at the point $z_0 \in Z$ and let $H : \Sigma Z \times I \to \Sigma Z$ be a homotopy between the identity mapping $\Sigma Z \to \Sigma Z$ and the constant one. Since Z is a compact space there exists a number $\varepsilon > 0$ such that no image by H of any sets with diameter less than ε contains both vertices v_0 and v_1.

Let Z' be a closed neighborhood of z_0 with diameter less than ε and such that the inclusion $Z' \hookrightarrow Z$ is homotopically nontrivial. Consider the inclusion $i : \Sigma Z' \hookrightarrow \Sigma Z$. The restriction of H onto $\Sigma Z' \times I$ is a homotopy, connecting i and the constant mapping. We can assume that the sets of points of $\Sigma Z'$ with the same τ' have diameter less than ε. Applying Lemma (4.1) we can assume also that H is a flat homotopy.

Let $A = h^{-1}(0)$ and $B = h^{-1}(1)$. We shall prove that then there exists a path $l : [0,1] \to I^2$ such that $l(0) = (\tau_1', 0), l(1) \in ((0 \times I) \cup (I \times 1) \cup (1 \times I))$ and $\operatorname{Im} l \cap (A \cup B) = \varnothing$. Let δ be the distance between closed sets A and B. Consider any natural triangulation T of the square I^2 with mesh less than $\frac{\delta}{4}$. Let P be the union of all simplexes of the T intersecting B. The regular neighborhood Q of polyhedron P in I^2 is a submanifold of I^2 (see, e.g. [11, Proposition (3.10)]), not intersecting A. The boundary of manifold Q is a discrete union of a finite number of simple closed curves. Denote by S one of this curves which contains the vertex $(1, 0)$ of I^2.

The curve S contains vertices of T which belong to $I \times 0$ or to $((0 \times I) \cup (I \times 1) \cup (1 \times I) \setminus \{(1, 0)\})$. Denote all of them by V_0 and V_1 respectively. Vertices V_0 and V_1, divide S into finite number of simple arcs. The number of arcs whose one end belongs to V_0 and other to V_1, is even. One of such arcs is a 1-simplex of T with vertex $(1, 0)$, which lies in $1 \times I$.

Therefore there exists the arc missing both A and B, one end of which belongs to V_0 and other to V_1. Natural parametrisation of this arc gives the desired path $l : [0, 1] \to I^2$.

Consider the cone $C(Z', \tau_1') = \{[z', \tau'] | [z', \tau'] \in \Sigma Z' \text{ and } \tau' \in [\tau_1', 1]\}$ and define a mapping $g : C(Z', \tau_1') \to \Sigma Z \setminus \{v_0, v_1\}$ by

$$g([z', \tau']) = H\left(\left[z', p_1 l\left(\frac{\tau' - \tau_1'}{1 - \tau_1'}\right)\right], p_2 l\left(\frac{\tau' - \tau_1'}{1 - \tau_1'}\right)\right).$$

Then because a cone is a contractible space, g maps the base $[Z', \tau_1']$ inessentially to $\Sigma Z \setminus \{v_0, v_1\}$. However, from the homotopical point of view, the restriction of g onto this base is the inclusion $Z' \hookrightarrow Z$ which is homotopically nontrivial. Contradiction. □

Corollary (4.2). *The suspension $\Sigma X_{\mathcal{P}}$ is noncontractible.*

Proof. By Proposition (3.6), the compactum $X_{\mathcal{P}}$ is not weakly contractible. So by Theorem (1.2), $\Sigma X_{\mathcal{P}}$ is noncontractible. □

Example (4.3). Let for every $i \in \mathbb{N}$, P_i be a finite noncontractible acyclic polyhedron. Let $Y = \bigvee_{i=1}^{\infty} P_i$ be the compact bouquet of P_i's. Although all the suspensions ΣP_i are contractible spaces, ΣY is noncontractible, by Theorem (1.2).

Question (4.4). Is the double suspension $\Sigma(\Sigma X_{\mathcal{P}})$ a noncontractible space?

Question (4.5). Does there exist a noncontractible locally contractible cell-like compactum?

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References

1. G. E. Bredon, *Sheaf Theory*, (Second Edition), Springer-Verlag, New York, 1997. CMP 98:04
2. J. H. Case and R. E. Chamberlin, *Characterization of tree-like continua*, Pacific J. Math. **10** (1960), 73–84. MR **22:**1868
3. R. J. Daverman and J. J. Walsh, *Examples of cell-like maps that are not shape equivalences*, Michigan Math. J. **30** (1983), 17–30. MR **85g:**57008

4. D. M. Hyman, *ANR divisors and absolute neighborhood contractibility*, Fund. Math. **62** (1968), 61–73. MR **37:**4771
5. U. H. Karimov, *An example of a space of trivial shape all of whose fine coverings are cyclic*, Dokl. Akad. Nauk SSSR **286**:3 (1986), 531–534 (in Russian); English transl. in Soviet Math. Dokl. **33** (1986), 113–117. MR **87i:**55016
6. R. C. Lacher, *Cell-like mappings*, I, Pacific J. Math. **30** (1969), 717–731. MR **40:**4941
7. R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin, 1977. MR **58:**28182
8. W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Interscience, New York, 1966. MR **34:**7617
9. S. Mardešić and J. Segal, *Shape Theory*: *The Inverse System Approach*, North-Holland, Amsterdam, 1982. MR **84b:**55020
10. J. van Mill and G. M. Reed, Eds., *Open Problems In Topology*, North-Holland, Amsterdam, 1990. MR **92c:**54001
11. C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, Ergebn. der Math. **69**, Springer-Verlag, Berlin, 1972. MR **50:**3236
12. H. Seifert and W. Threlfall, *A Textbook of Topology*, Pure Appl. Math. **89**, Academic Press, New York, 1980. MR **82b:**55001
13. L. C. Siebenmann, *Chapman's classification of shapes. A proof using collapsing*, Manuscripta Math. **16** (1975), 373–384. MR **55:**4185

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